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FOREWORD

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Algebraic Fractions Applied to Logarithmic Functions

By Douglas G. Anderson
Associate with
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Franklin, Michigan

ABSTRACT

In this paper a method of interpolation is presented which permits abbreviation of six-place tables to the length of four-place tables without loss of accuracy. Tables adapted to the use of the method are appended.

Also, a very brief formula is developed for use by a binary computer in direct computation of natural logarithms. The formula is correct within one unit in the tenth decimal place.

INTRODUCTION

Numerical computation by means of logarithms has dwindled since calculating machines have become common. Such machines, however, offer slight advantage for trigonometric computation, which requires frequent reference to tables and extraction of roots. It appears that substantial savings of labor can be realized by shortening the tables.

Such condensation requires an improved method of interpolation, since the principal reason for the bulk of the usual tables is the need to avoid serious loss of accuracy in interpolation.

PLAN FOR THE TABLES

To attain the greatest convenience for trigonometric computation, each of the essential tables will appear on one double-page. The table of log N will give mantissas of the three-digit integers. The tables of log sin A and log tan A will have a tabular interval of ten minutes. An interval of .1 degree has much merit, but the number of entries required would crowd a page the size of those in this journal. The interval chosen is most convenient for work in degrees and minutes.

The table of log sines will extend to 90°, and thus will contain the necessary log cosines. The table of log tangents will extend only to 45°, since $\log \tan A = -\log \cot A = -\log \tan(90^\circ - A)$.

It is anticipated that a ten-inch slide-rule will be used for interpolation. The formulas, however, will be sufficiently accurate to justify working to six places even where this is beyond the ability of the slide-rule.

Linear interpolation will be used for angles larger than 30°, because it is slightly easier, saves space in the tables, and is sufficiently accurate.

The familiar functions, S and T, are given for angles less than 2°, with the angles expressed in minutes.

INTERPOLATION FOR LOGARITHMS OF NUMBERS

Let N be a three-digit integer, and let n be a number less than one in absolute magnitude. It will be assumed throughout this discussion that the base of logarithms is 10.

$$\log (N + n) = \log N + \log \left(1 + \frac{n}{N}\right). \tag{1}$$

With the base 10, any number with the same sequence of digits has the same mantissa. The characteristic may be found by inspection, the mantissa of N may be read from the table, and the problem reduces to finding the value of

$$d_n = \log\left(1 + \frac{n}{N}\right), \tag{2}$$

the interpolation function for numbers.

From two well-known[1] series, both convergent when $x^2 < 1$, we obtain

$$\log(1 + x) = (\log e)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) \tag{3}$$

and

$$\frac{x \log e}{1 + \frac{1}{2}x} = (\log e)\left(x - \frac{x^2}{2} + \frac{x^3}{4} - \frac{x^4}{8} + \cdots\right). \tag{4}$$

Therefore

$$\log(1 + x) = \frac{x \log e}{1 + \frac{1}{2}x} + E, \tag{5}$$

where the first term on the right is an approximation to d, and the error E, is

$$E = (\log e)\left(\frac{1}{12}x^3 - \frac{1}{8}x^4 + \frac{11}{80}x^5 - \cdots\right). \tag{6}$$

Setting $x_n = \frac{n}{N}$ in eq. 5, and substituting in eq. 2,

$$d_n = \frac{n \log e}{N + \frac{1}{2}n} + E_n .$$

Since $x_n < \pm .01$, $E_n < \pm .000\ 000\ 04$, completely negligible for a six-place table.

INTERPOLATION FOR LOGARITHMS OF TRIGONOMETRIC FUNCTIONS

Let A be a tabulated angle not less than 1^0, and let a be an angle not larger than $\pm 5'$.

$$\begin{aligned}\sin(A + a) &= \sin A \cos a + \cos A \sin a \\ &= \sin A \cos a\,(1 + \cot A \tan a) .\end{aligned}$$

Therefore

$$\log \sin(A + a) = \log \sin A + \log \cos a + \log (1 + \cot A \tan a). \tag{7}$$

Setting $x_s = \cot A \tan a$, in eq. 5, gives

$$\log(1 + \cot A \tan a) = \frac{\tan a \log e}{\tan A + \frac{1}{2}\tan a} + E_s . \tag{8}$$

For small angles expressed in minutes, $\tan a = \frac{a}{3438}$. Therefore eq. 8 may be written

$$d_s = \frac{a \log e}{3438 \tan A + \frac{1}{2}a} + E_s . \tag{9}$$

Also

$$\tan(A + a) = \frac{\tan A + \tan a}{1 - \tan A \tan a} = \tan A \left(\frac{1 + \cot A \tan a}{1 - \tan A \tan a}\right)$$

$$= \tan A \left[1 + \frac{(\tan A + \cot A)\tan a}{1 - \tan A \tan a}\right]$$

$$= \tan A \left(1 + \frac{\tan a}{\sin A \cos A - \sin^2 A \tan a}\right) .$$

Therefore

$$\log \tan(A + a) = \log \tan A + \log(1 + x_t) , \tag{10}$$

where $x_t = \dfrac{\tan a}{\sin A \cos A - \sin^2 A \tan a}$. By eq. 5, where $x = x_t$,

$$\log(1 + x_t) = \frac{x_t \log e}{1 + \frac{1}{2} x_t} + E_t$$

$$= \frac{\tan a \log e}{\sin A \cos A + \frac{1}{2}(1 - 2\sin^2 A)\tan a} + E_t$$

$$= \frac{\tan a \log e}{\frac{1}{2} \sin 2A + \frac{1}{2} \tan a \cos 2A} + E_t$$

$$= \frac{a \log e}{1719 \sin 2A + \frac{1}{2} a \cos 2A} + E_t$$

$$= \frac{a \log e}{1719 \sin 2A + \frac{1}{2} a} + E_c + E_t . \tag{11}$$

The last step above was taken to separate the variables in the denominator of the interpolation function. The term thus introduced is, since $\log e = .4343$,

$$E_c = \frac{.4343\, a^2 (1 - \cos 2A)}{2(1719 \sin 2A + \frac{1}{2} a)(1719 \sin 2A + \frac{1}{2} a \cos 2A)} ,$$

which is closely approximated by

$$E_c = \frac{.4343\, a^2 (1 - \cos 2A)}{2(1719)^2 \sin 2A} = \frac{.4343\, a^2}{(3438)^2} \sec^2 A . \tag{12}$$

Then eq. 10 may be approximated by

$$\log \tan (A + a) = \log \tan A + \frac{.4343\, a}{1719 \sin 2A + \frac{1}{2} a} , \tag{13}$$

where E_t and E_c are omitted.

Thus the interpolation function for either log sine or log tangent may be represented by the same form,

$$d = \frac{.4343\, a}{f(A) + \frac{1}{2} a} , \tag{14}$$

which represents the interpolation function for the log of the number $(A + a)$.

ACCURACY OF INTERPOLATION FUNCTION

Eq. 7 may be written

$$\log \sin (A + a) = \log \sin A + \frac{.4343\ a}{3438 \tan A + \frac{1}{2} a}, \qquad (15)$$

where log cos a and E_s are omitted. This omission introduces error. The sum of log cos a and E_s has its greatest absolute value at $a = -5'$. $|E_s|$ is greatest when $|A|$ is smallest. Thus, at $A = 1^\circ\ 10'$, and $a = -5'$, $x_s = -.0714$, and

$$\log \cos a + E_s = -.000\ 000\ 46 - .000\ 014\ 8$$
$$= -.000\ 015\ 3 .$$

When $A = 2^\circ\ 10'$, a similar computation yields $-.000\ 002\ 6$.

Thus, if $A > 2^\circ$, the error in log sine as computed by eq. 15 is less than $\pm .000\ 002\ 6$. For large angles, the error is less than .000 000 5.

Omission of E_t and E_c introduces error to eq. 13. Their sum is greater when $a = 5'$. When $a = 5'$ and $A = 1^\circ$, $x_t = .0833$, and

$$E_t + E_c = .000\ 018\ 4 + .000\ 000\ 92$$
$$= .000\ 019\ 3 .$$

When $A = 2^\circ$, a similar computation yields .000 003 4. For large angles, E_t is completely negligible, but E_c increases with A. At 30°, $\sec^2 A = \frac{4}{3}$, and $E_c = .000\ 001\ 2$. Thus, when $2^\circ \leq A \leq 30^\circ$, the error in log tangent as computed by eq. 13 is less than $\pm .000\ 003\ 4$.

The errors computed above can be further reduced. E_t and E_s become less than .000 000 1 when $A = 6^\circ$. When $A < 6^\circ$, $x_s = x_t = \frac{a}{A}$, very nearly. Also, between 2° and 6°, $d = .4343 \frac{a}{A}$, approximately, and the major portion of E_s or E_t is $.0362 (\frac{a}{A})^3$. Assuming a small adjustment, Δ, in the denominator of d, to cancel this error,

$$\frac{a \log e}{A - \Delta} - \frac{a \log e}{A} = \frac{\log e}{12} (\frac{a}{A})^3 ,$$

from which $$\Delta = \frac{a^2}{12\,A}\,. \tag{16}$$

The best choice for a in eq. 16 is that which minimizes the total error over the whole range of a. A study of the errors at $A = 2^\circ$ leads to

$$d_s = \frac{.4343\,a}{Z + \frac{1}{2}a} \tag{17}$$

and

$$d_t = \frac{.4343\,a}{R + \frac{1}{2}a}\,, \tag{18}$$

in which

$$Z = 3438 \tan A - \frac{.000\,510}{\tan A} \tag{19}$$

and

$$R = 1719 \sin 2A - \frac{.001\,21}{\sin 2A} \tag{20}$$

Then, the error in log sine does not exceed $\pm$ .000 001 when $A > 2^\circ$. Also, the error in log tangent does not exceed $\pm$ .000 001 2, from 2° to 30°.

SLIDE-RULE ERROR

A study of slide-rule accuracy indicates a probable error of .07% in a three-factor product as obtained by careful use of a ten-inch slide-rule. Accepting .10% as a conservative estimate, the probable error in evaluation of d is

$$E_{SR} = \pm .000\,43\,\frac{a}{f(A) + \frac{1}{2}a}\,.$$

This error is in general much larger than all other errors. It is equivalent to an angular probable error of $\pm$ .0010 a, which does not exceed .005 minute, or three-tenths of a second. This, in turn, is less than the probable angular error due to rounding off to six decimal places in the table of log sines for angles greater than 52°.

LINEAR INTERPOLATION

For angles over 30°, A and R are omitted from the tables, and tabular differences are given. This saves space. Also, use of the linear interpolation function

$$d = \frac{a}{10'} D ,$$

where D is the tabular difference, avoids the mental addition of $f(A) + \frac{1}{2} a$. The error does not exceed

$$\log \sin 30^\circ 5' - \tfrac{1}{2} (\log \sin 30^\circ 10' + \log \sin 30^\circ) = .000\ 001\ 8 .$$

SMALL ANGLES

As A decreases below 2°, the interpolative errors rapidly increase. But, for small angles, sin A and tan A are very nearly proportional to A. Therefore the functions $S = \frac{\sin A}{A}$ and $T = \frac{\tan A}{A}$ are slowly varying functions of A, for which linear interpolation is accurate and easy.

These functions have been frequently employed[2], in the equations

$$\log \sin (A + a) = \log (A + a) + S(A + a)$$

and

$$\log \tan (A + a) = \log (A + a) + T(A + a) .$$

It has been most common to express (A + a) in seconds, but where the tabular interval in the main tables is ten minutes, it is better to express (A + a) in minutes. The special table appended to this paper gives S and T to six places without interpolation.

INVERSE INTERPOLATION

Solving the equation

$$d_n = \frac{.4343\ n}{N + \frac{1}{2} n}$$

for n gives the rule for inverse interpolation

$$n = \frac{N\, d_n}{.4343 - \frac{1}{2} d_n}. \tag{21}$$

Similarly,

$$a = \frac{Z\, d_s}{.4343 - \frac{1}{2} d_s} \tag{22}$$

and

$$a = \frac{R\, d_t}{.4343 - \frac{1}{2} d_t} \tag{23}$$

These formulas are easily memorized.

Inverse interpolation for small angles requires that an approximate value for the angle be known, in order to read S or T from the table. This approximation may readily be found on the slide-rule. Then

$$\log(A + a) = \log \sin(A + a) - S, \tag{24}$$

and

$$\log(A + a) = \log \tan(A + a) - T. \tag{25}$$

USE OF THE TABLES

It is assumed that the user is familiar with logarithms. These directions will deal only with interpolation.

1. Given a number, to find its logarithm:
   Disregarding the actual decimal point, think of the given number as a number between 100 and 1000 with the given sequence of digits. From this number, subtract N, the nearest three-digit integer, to obtain n, a decimal fraction between - 0.5 and 0.5. Read log N from the table, and compute

$$\log (N + n) = \log N + \frac{.4343\, n}{N + \frac{1}{2} n}.$$

   The given number has the same mantissa as N + n.

2. Given a logarithm, to find the corresponding number:
Find the nearest tabulated mantissa, log N, and read N. Subtract log N from log (N + n), the given mantissa. (The resulting number, d_n, may be either positive or negative.) Then

$$N + n = N + \frac{N\, d_n}{.4343 - \frac{1}{2} d_n}.$$

3. Given an angle between 2^0 and 30^0, to find its log sine:
From the given angle subtract A, the nearest tabulated angle, to obtain a, an angle between - 5′ and 5′. From the table, read log sin A, and Z. Compute the required log sine,

$$\log \sin(A + a) = \log \sin A + \frac{.4343\, a}{Z + \frac{1}{2} a}.$$

4. Given a log sine, to find the corresponding angle, between 2^0 and 30^0:
Find the nearest tabulated value, log sin A, and read A. Subtract log sin A from the given value, log sin(A + a), to obtain d_s. Compute the required angle,

$$A + a = A + \frac{Z\, d_s}{.4343 - \frac{1}{2} d_s},$$

where a is expressed in minutes.

5. Given an angle between 2^0 and 30^0, to find its log tangent:
From the given angle subtract A, the nearest tabulated angle, to obtain a, an angle between - 5′ and 5′. From the table, read log tan A, and R. Compute the required log tangent,

$$\log \tan(A + a) = \log \tan A + \frac{.4343\, a}{R + \frac{1}{2} a}.$$

6. Given a log tangent, to find the corresponding angle, between 2^0 and 30^0:
Find the nearest tabulated value, log tan A, and read A. Subtract log tan A from the given value, log tan(A + a), to obtain d_t. Compute the required angle,

$$A + a = A + \frac{R\, d_t}{.4343 - \frac{1}{2} d_t},$$

where a is expressed in minutes.

7. Given a small angle expressed in minutes, to find its log sine:
Read S from the table of S and T, for the given angle, A. Compute log A by rule 1. Then

$$\log \sin A = \log A + S .$$

8. Given a small angle expressed in minutes, to find its log tangent:
Read T from the table of S and T, for the given angle, A. Compute log A by rule 1. Then

$$\log \tan A = \log A + T .$$

9. Given the log sine of a small angle, to find the angle:
Find an approximate value for the angle, A, by use of the slide-rule. With this value, find S from the table of S and T. Compute

$$\log A = \log \sin A - S .$$

Find A, expressed in minutes, by rule 2.

10. Given the log tangent of a small angle, to find the angle:
Find an approximate value for the angle, A, by use of the slide-rule. With this value, find T from the table of S and T. Compute

$$\log A = \log \tan A - T .$$

Find A, expressed in minutes, by rule 2.

AN ACCURATE APPROXIMATION TO $\log_e X$

Several methods have been developed for approximating $\log_e x$ on high-speed computers. The more recent ones are modifications of a method due to I. J. Cherry[3], which makes use of the relation

$$\log_e x = (I + \log_2 f)\log_e 2 ,$$

where $x = 2^I f$, and $\frac{1}{2} \leq f < 1$. This may also be written

$$\log_e x = I \log_e 2 + \log_e f .$$

In binary computers using "floating point" arithmetic, a number is expressed in terms of I and f. Thus the problem is to express $\log_e f$, briefly and accurately. The series[4]

$$\log_e \frac{(1 + z)}{(1 - z)} = 2z + (2/3)z^3 + (2/5)z^5 + (2/7)z^7 + \cdots \qquad (26)$$

is a good starting point. Setting $y = \frac{1 + z}{1 - z}$, as y varies through the range from $\frac{1}{2}\sqrt{2}$ to $\sqrt{2}$, z varies through the range from $-(3 - 2\sqrt{2})$ to $+(3 - 2\sqrt{2})$, which is smaller. Setting $y = \sqrt{2}\, f$, we obtain $z = \frac{f - \frac{1}{2}\sqrt{2}}{f + \frac{1}{2}\sqrt{2}}$, and

$$\log_e x = (1 - \tfrac{1}{2})\log_e 2 + \log_e y . \qquad (27)$$

The series for $\log_e y$ can be closely approximated by

$$L = \left(\frac{a}{1 - b z^2} + \frac{c}{1 - d z^2}\right) z = (a + c)z + (ab + cd)z^3 + (ab^2 + cd^2)z^5 + (ab^3 + cd^3)z^7 + \cdots . \qquad (28)$$

Initially, the constants may be chosen to make the first four terms of eq. 28 agree with the corresponding terms of eq. 26. Then $a = 1 + (1/18)\sqrt{30}$, $c = 1 - (1/18)\sqrt{30}$, $b = \frac{3}{7} - (2/35)\sqrt{30}$, and $d = \frac{3}{7} + (2/35)\sqrt{30}$.

Comparison of higher terms of the two series shows that the error is

$$\log y - L = .01161\, z^9 + .02578\, z^{11} + .0382\, z^{13} + \cdots . \qquad (29)$$

When $z = 3 - 2\sqrt{2} = .17157$, the error is 15.98×10^{-10}.

This error can be greatly reduced by small adjustments in a, b, c and d which leave the first three terms of the series in eq. 28 unchanged. Thus

$$(a + \Delta a) + (c + \Delta c) = a + c .$$

$$\therefore \Delta a = -\Delta c . \qquad (30)$$

Also

$$(a - \Delta c)(b + \Delta b) + (c + \Delta c)(d + \Delta d) = ab + cd .$$

$$\therefore \Delta c = -\frac{a\,\Delta b + c\,\Delta d}{(d + \Delta d) - (b + \Delta b)} . \qquad (31)$$

And

$$(a - \Delta c)(b + \Delta b)^2 + (c + \Delta c)(d + \Delta d)^2 = ab^2 + cd^2 .$$

$$\therefore \Delta c = -\frac{a(2b + \Delta b)\,\Delta b + c(2d + \Delta d)\,\Delta d}{(d + \Delta d)^2 - (b + \Delta b)^2} . \qquad (32)$$

Equating these two expressions for Δc, we obtain

$$\Delta b = \frac{c(d - b)\,\Delta d}{a(d - b) + (c + a)\,\Delta d}. \tag{33}$$

These equations give exact values for all the changes, as fractions of the change in d, such that no error will occur in the first three terms of L.

To determine the change in d, we expand the higher terms of L, neglecting powers of Δd higher than the first.

$$\begin{aligned}\Delta L = {} & [(a - \Delta c)(b + \Delta b)^3 + (c + \Delta c)(d + \Delta d)^3 - (ab^3 + cd^3)]z^7 \\ & + [(a - \Delta c)(b + \Delta b)^4 + (c + \Delta c)(d + \Delta d)^4 - (ab^4 + cd^4)]z^9 \\ & + \cdots . \end{aligned} \tag{34}$$

But, to a first-degree approximation, $\Delta b = \frac{c}{a}\,\Delta d$ and $\Delta c = -\frac{2c}{d - b}\,\Delta d$. Therefore, to a first-degree approximation,

$$\begin{aligned}\frac{\Delta L}{\Delta d} &= [3(d^2 + b^2) - \frac{2}{d - b}(d^3 - b^3)]c\,z^7 \\ &\quad + [4(d^3 + b^3) - \frac{2}{d - b}(d^4 - b^4)]c\,z^9 + \cdots \\ &= .2726\,z^7 + .4673\,z^9 + .5\,5\,42\,z^{11} + \cdots .\end{aligned}$$

The error equation is now

$$E = \log_e y - (L + \Delta L).$$

In expanding this equation it is convenient to replace z by $(3 - 2\sqrt{2})r = .17157\,r$. Then

$$\begin{aligned}E = 43.76 \times 10^{-10} [& -272.6\,\Delta d\,r^7 + (.3418 - 13.77\,\Delta d)\,r^9 \\ & + (.02234 - .480\,\Delta d)\,r^{11} + \cdots].\end{aligned}$$

Evaluating this for two values of Δd, say .0012 and .0010, and for several values of r, including r = 1, the most favorable value for Δd may be found graphically, as shown in the figure. When $\Delta d = .001185$ the maximum absolute error is 1.09×10^{-10}.

All that remains is to calculate the final values of the constants, and to put the expression in the most convenient form. This is

$$\log_e y = \left[\frac{\frac{a}{b}}{\frac{1}{b} - z^2} + \frac{\frac{c}{d}}{\frac{1}{d} - z^2}\right] z .$$

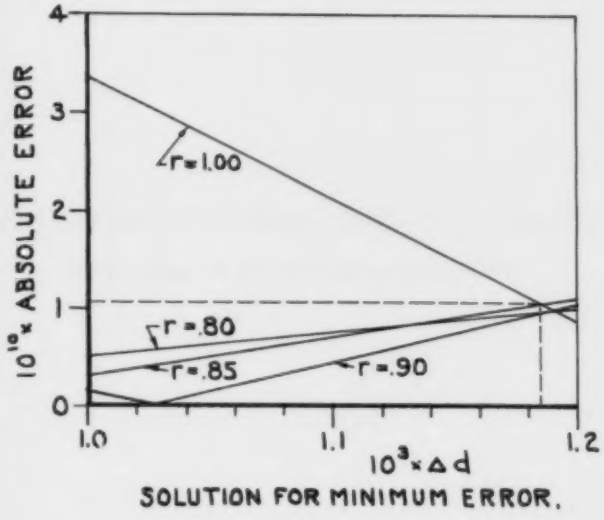

Figure 1

The final values are

$$\frac{a}{b} = 11\ .245\ 464\ 629\ 1 \qquad \frac{c}{d} = .933\ 140\ 917\ 51$$

$$\frac{1}{b} = 8.604\ 566\ 434\ 81 \qquad \frac{1}{d} = 1.346\ 364\ 803\ 34\ .$$

ACKNOWLEDGMENT

The author is indebted to Miss Joyce Yeomans of the General Motors Technical Center for her work in programming the computation which made it possible to publish the appended tables.

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LOGARITHMS OF NUMBERS, SINES AND TANGENTS, WITH FUNCTIONS FOR INTERPOLATION.

Copyright 1960, Douglas G. Anderson.

COMMON LOGARITHMS.

N	0	1	2	3	4	5	6	7	8	9
10	000 000	004 321	008 600	012 837	017 033	021 189	025 306	029 384	033 424	037 426
11	041 393	045 323	049 218	053 078	056 905	060 698	064 458	068 186	071 882	075 547
12	079 181	082 785	086 360	089 905	093 422	096 910	100 371	103 804	107 210	110 590
13	113 943	117 271	120 574	123 852	127 105	130 334	133 539	136 721	139 879	143 015
14	146 128	149 219	152 288	155 336	158 362	161 368	164 353	167 317	170 262	173 186
15	176 091	178 977	181 844	184 691	187 521	190 332	193 125	195 900	198 657	201 397
16	204 120	206 826	209 515	212 188	214 844	217 484	220 108	222 716	225 309	227 887
17	230 449	232 996	235 528	238 046	240 549	243 038	245 513	247 973	250 420	252 853
18	255 273	257 679	260 071	262 451	264 818	267 172	269 513	271 842	274 158	276 462
19	278 754	281 033	283 301	285 557	287 802	290 035	292 256	294 466	296 665	298 853
20	301 030	303 196	305 351	307 496	309 630	311 754	313 867	315 970	318 063	320 146
21	322 219	324 282	326 336	328 380	330 414	332 438	334 454	336 460	338 456	340 444
22	342 423	344 392	346 353	348 305	350 248	352 183	354 108	356 026	357 935	359 835
23	361 728	363 612	365 488	367 356	369 216	371 068	372 912	374 748	376 577	378 398
24	380 211	382 017	383 815	385 606	387 390	389 166	390 935	392 697	394 452	396 199
25	397 940	399 674	401 401	403 121	404 834	406 540	408 240	409 933	411 620	413 300
26	414 973	416 641	418 301	419 956	421 604	423 246	424 882	426 511	428 135	429 752
27	431 364	432 969	434 569	436 163	437 751	439 333	440 909	442 480	444 045	445 604
28	447 158	448 706	450 249	451 786	453 318	454 845	456 366	457 882	459 392	460 898
29	462 398	463 893	465 383	466 868	468 347	469 822	471 292	472 756	474 216	475 671
30	477 121	478 566	480 007	481 443	482 874	484 300	485 721	487 136	488 551	489 958
31	491 362	492 760	494 155	495 544	496 930	498 311	499 687	501 059	502 427	503 791
32	505 150	506 505	507 856	509 203	510 545	511 883	513 218	514 548	515 874	517 196
33	518 514	519 826	521 136	522 444	523 746	525 045	526 339	527 630	528 917	530 200
34	531 479	532 754	534 026	535 294	536 558	537 819	539 076	540 329	541 579	542 825
35	544 068	545 307	546 543	547 775	549 003	550 228	551 450	552 668	553 883	555 094
36	556 303	557 507	558 709	559 907	561 101	562 293	563 481	564 666	565 848	567 026
37	568 202	569 374	570 543	571 709	572 872	574 031	575 188	576 341	577 492	578 639
38	579 784	580 925	582 063	583 197	584 331	585 461	586 587	587 711	588 832	589 950
39	591 065	592 177	593 286	594 393	595 496	596 597	597 695	598 791	599 883	600 973
40	602 060	603 144	604 226	605 305	606 381	607 455	608 526	609 594	610 660	611 723
41	612 784	613 842	614 897	615 950	617 000	618 048	619 093	620 136	621 176	622 214
42	623 249	624 282	625 312	626 340	627 366	628 389	629 410	630 428	631 444	632 457
43	633 468	634 477	635 484	636 488	637 490	638 489	639 486	640 481	641 474	642 465
44	643 453	644 439	645 422	646 404	647 383	648 360	649 335	650 308	651 278	652 246
45	653 212	654 177	655 138	656 098	657 056	658 011	658 965	659 916	660 865	661 813
46	662 758	663 701	664 642	665 581	666 518	667 453	668 386	669 317	670 246	671 173
47	672 098	673 021	673 942	674 861	675 776	676 694	677 607	678 518	679 426	680 336
48	681 241	682 145	683 047	683 947	684 845	685 742	686 636	687 529	688 420	689 309
49	690 196	691 081	691 965	692 847	693 727	694 605	695 482	696 356	697 229	698 100

COMMON LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9
50	698 970	699 838	700 704	701 568	702 431	703 291	704 151	705 008	705 864	706 718
51	707 570	708 421	709 270	710 117	710 963	711 807	712 650	713 491	714 330	715 167
52	716 003	716 838	717 671	718 502	719 331	720 159	720 986	721 811	722 634	723 456
53	724 276	725 095	725 912	726 727	727 541	728 354	729 165	729 974	730 782	731 589
54	732 394	733 197	733 999	734 800	735 599	736 397	737 193	737 987	738 781	739 572
55	740 363	741 152	741 939	742 725	743 510	744 293	745 075	745 855	746 634	747 412
56	748 188	748 963	749 736	750 508	751 279	752 048	752 816	753 583	754 348	755 112
57	755 875	756 636	757 396	758 155	758 912	759 668	760 422	761 176	761 928	762 679
58	763 428	764 176	764 923	765 669	766 413	767 156	767 898	768 638	769 377	770 115
59	770 852	771 587	772 322	773 055	773 786	774 517	775 246	775 974	776 701	777 427
60	778 151	778 874	779 596	780 317	781 037	781 755	782 473	783 189	783 904	784 617
61	785 330	786 041	786 751	787 460	788 168	788 875	789 581	790 285	790 988	791 691
62	792 392	793 092	793 790	794 488	795 185	795 880	796 574	797 268	797 960	798 651
63	799 341	800 029	800 717	801 404	802 089	802 774	803 457	804 139	804 821	805 501
64	806 180	806 858	807 535	808 211	808 886	809 560	810 233	810 904	811 575	812 245
65	812 913	813 581	814 248	814 913	815 578	816 241	816 904	817 565	818 226	818 885
66	819 544	820 201	820 858	821 514	822 168	822 822	823 474	824 126	824 776	825 426
67	826 075	826 723	827 369	828 015	828 660	829 304	829 947	830 589	831 230	831 870
68	832 509	833 147	833 784	834 421	835 056	835 691	836 324	836 957	837 588	838 219
69	838 849	839 478	840 106	840 733	841 359	841 985	842 609	843 233	843 855	844 477
70	845 098	845 718	846 337	846 955	847 573	848 189	848 805	849 419	850 033	850 648
71	851 258	851 870	852 480	853 090	853 698	854 306	854 913	855 519	856 124	856 729
72	857 332	857 935	858 537	859 138	859 739	860 338	860 937	861 534	862 131	862 728
73	863 323	863 917	864 511	865 104	865 696	866 287	866 878	867 467	868 056	868 644
74	869 232	869 818	870 404	870 989	871 573	872 156	872 739	873 321	873 902	874 482
75	875 061	875 640	876 218	876 795	877 371	877 947	878 522	879 096	879 669	880 242
76	880 814	881 385	881 955	882 525	883 093	883 661	884 229	884 795	885 361	885 926
77	886 491	887 054	887 617	888 179	888 741	889 302	889 862	890 421	890 980	891 537
78	892 095	892 651	893 207	893 762	894 316	894 870	895 423	895 975	896 526	897 077
79	897 627	898 176	898 725	899 273	899 821	900 367	900 913	901 458	902 003	902 547
80	903 090	903 632	904 174	904 716	905 256	905 796	906 335	906 874	907 411	907 949
81	908 485	909 021	909 556	910 091	910 624	911 158	911 690	912 222	912 753	913 284
82	913 814	914 343	914 872	915 400	915 927	916 454	916 980	917 506	918 030	918 559
83	919 078	919 601	920 123	920 645	921 166	921 686	922 206	922 725	923 244	923 762
84	924 279	924 796	925 312	925 828	926 342	926 857	927 370	927 883	928 396	928 908
85	929 419	929 930	930 440	930 949	931 458	931 966	932 474	932 981	933 487	933 993
86	934 498	935 003	935 507	936 011	936 514	937 016	937 518	938 019	938 520	939 020
87	939 519	940 018	940 516	941 014	941 511	942 008	942 504	943 000	943 495	943 989
88	944 483	944 976	945 469	945 961	946 452	946 945	947 434	947 924	948 413	948 902
89	949 390	949 878	950 365	950 851	951 338	951 823	952 308	952 792	953 276	953 760
90	954 242	954 725	955 207	955 688	956 168	956 649	957 128	957 607	958 086	958 564
91	959 041	959 518	959 995	960 471	960 946	961 421	961 895	962 369	962 843	963 316
92	963 788	964 260	964 731	965 202	965 672	966 142	966 611	967 080	967 548	968 016
93	968 483	968 950	969 416	969 882	970 347	970 812	971 276	971 740	972 203	972 666
94	973 128	973 590	974 051	974 512	974 972	975 432	975 891	976 350	976 808	977 266
95	977 724	978 181	978 637	979 093	979 548	980 003	980 458	980 912	981 366	981 819
96	982 271	982 723	983 175	983 626	984 077	984 527	984 977	985 426	985 875	986 324
97	986 772	987 219	987 666	988 113	988 559	989 005	989 450	989 895	990 339	990 783
98	991 226	991 669	992 111	992 554	992 995	993 436	993 877	994 317	994 757	995 196
99	995 635	996 074	996 512	996 949	997 386	997 823	998 259	998 695	999 131	999 565

10 PLUS LOG SINE A, WITH Z FUNCTION.

A	0′	10′	20′	30′	40′	50′	60′	
0°	Use S=6.46372, Or For Greater Accuracy, See Special Table Of S & T.							89
1	8.241 855 39. 97	8.308 794 69. 98	8.366 777 99. 99	8.417 919 90. 00	8.463 665 100. 01	8.505 045 110. 02	8.542 819 120. 03	88
2	8.542 819 120. 03	8.577 566 130. 05	8.609 734 140. 07	8.639 680 150. 09	8.667 689 160. 11	8.695 998 170. 13	8.718 800 180. 16	87
3	8.718 800 180. 16	8.742 259 190. 19	8.764 511 200. 22	8.785 675 210. 26	8.805 852 220. 30	8.825 190 230. 35	8.843 585 240. 39	86
4	8.843 585 240. 39	8.861 283 250. 45	8.878 285 260. 50	8.894 643 270. 56	8.910 404 280. 63	8.925 605 290. 70	8.940 296 300. 78	85°
5°	8.940 296 300. 78	8.954 499 310. 86	8.968 249 320. 94	8.981 573 331. 03	8.994 497 341. 13	9.007 044 351. 23	9.019 235 361. 34	84
6	9.019 235 361. 34	9.031 089 371. 46	9.042 625 381. 58	9.053 859 391. 71	9.064 806 401. 84	9.075 480 411. 99	9.085 894 422. 14	83
7	9.085 894 422. 14	9.096 062 432. 29	9.105 992 442. 46	9.115 698 452. 63	9.125 187 462. 80	9.134 470 472. 99	9.143 555 483. 19	82
8	9.143 555 483. 19	9.152 451 493. 39	9.161 184 503. 61	9.169 702 513. 83	9.178 072 524. 06	9.186 280 534. 29	9.194 332 544. 53	81
9	9.194 332 544. 53	9.202 234 554. 80	9.209 992 565. 07	9.217 609 575. 35	9.225 092 585. 64	9.232 444 595. 93	9.239 670 606. 24	80°
10°	9.239 670 606. 24	9.246 775 616. 56	9.253 761 626. 89	9.260 633 637. 23	9.267 395 647. 58	9.274 049 657. 95	9.280 599 668. 32	79
11	9.280 599 668. 32	9.287 048 678. 71	9.293 399 689. 11	9.299 655 699. 52	9.305 819 709. 94	9.311 893 720. 38	9.317 879 730. 82	78
12	9.317 879 730. 82	9.323 780 741. 29	9.329 599 751. 76	9.335 337 762. 25	9.340 996 772. 75	9.346 579 783. 26	9.352 088 793. 80	77
13	9.352 088 793. 80	9.357 524 804. 36	9.362 889 814. 90	9.368 185 825. 47	9.373 414 836. 06	9.378 577 846. 67	9.383 675 857. 28	76
14	9.383 675 857. 28	9.388 711 867. 92	9.393 685 878. 57	9.398 600 889. 23	9.403 455 899. 91	9.408 254 910. 61	9.412 996 921. 32	75°
15°	9.412 996 921. 32	9.417 684 932. 05	9.422 318 942. 80	9.426 899 953. 56	9.431 429 964. 34	9.435 908 975. 15	9.440 338 985. 96	74
16	9.440 338 985. 96	9.444 720 996. 80	9.449 054 1007. 6	9.453 342 1018. 5	9.457 584 1029. 4	9.461 782 1040. 3	9.465 935 1051. 2	73
17	9.465 935 1051. 2	9.470 046 1062. 2	9.474 115 1073. 1	9.478 142 1084. 1	9.482 128 1095. 1	9.486 075 1106. 2	9.489 982 1117. 2	72
18	9.489 982 1117. 2	9.493 851 1128. 3	9.497 682 1139. 4	9.501 476 1150. 5	9.505 234 1161. 6	9.508 956 1172. 8	9.512 642 1184. 0	71
19	9.512 642 1184. 0	9.516 294 1195. 2	9.519 911 1206. 4	9.523 495 1217. 6	9.527 046 1228. 9	9.530 565 1240. 2	9.534 052 1251. 5	70°
20°	9.534 052 1251. 5	9.537 507 1262. 9	9.540 931 1274. 2	9.544 325 1285. 6	9.547 689 1297. 0	9.551 024 1308. 5	9.554 329 1320. 0	69
21	9.554 329 1320. 0	9.557 606 1331. 4	9.560 855 1343. 0	9.564 075 1354. 5	9.567 269 1366. 1	9.570 435 1377. 7	9.573 575 1389. 3	68
22	9.573 575 1389. 3	9.576 689 1401. 0	9.579 777 1412. 6.	9.582 840 1424. 3	9.585 877 1436. 1	9.588 890 1447. 8	9.591 878 1459. 6	67
23	9.591 878 1459. 6	9.594 842 1471. 3	9.597 783 1483. 2	9.600 700 1495. 2	9.603 594 1507. 1	9.606 465 1519. 1	9.609 313 1531. 0	66
24	9.609 313 1531. 0	9.612 140 1543. 0	9.614 944 1555. 1	9.617 727 1567. 1	9.620 488 1579. 2	9.623 229 1591. 4	9.625 948 1603. 6	65°
25°	9.625 948 1603. 6	9.628 647 1615. 8	9.631 326 1628. 0	9.633 984 1640. 2	9.636 623 1652. 5	9.639 242 1664. 9	9.641 842 1677. 3	64
26	9.641 842 1677. 2	9.644 423 1689. 7	9.646 984 1702. 1	9.649 527 1714. 6	9.652 052 1727. 1	9.654 558 1739. 6	9.657 047 1752. 2	63
27	9.657 047 1752. 2	9.659 517 1764. 8	9.661 970 1777. 5	9.664 406 1790. 2	9.666 824 1802. 9	9.669 225 1815. 7	9.671 609 1828. 5	62
28	9.671 609 1828. 5	9.673 977 1841. 4	9.676 328 1854. 3	9.678 663 1867. 2	9.680 982 1880. 2	9.683 284 1893. 2	9.685 571 1906. 3	61
29	9.685 571 1906. 3	9.687 843 1919. 4	9.690 098 1932. 6	9.692 339 1945. 8	9.694 564 1959. 0	9.696 775 1972. 3	9.698 970 1985. 5	60°
	60′	50′	40′	30′	20′	10′	0	B

10 PLUS LOG COSINE B, WITH Z FUNCTION.

10 PLUS LOG SINE A. WITH DIFFERENCES.

A	0′		10′		20′		30′		40′		50′		60′		
30°	9.698 9770	2181	701 1951	2166	703 8117	2152	706 4283	2138	709 0421	2123	711 6544	2110	9.714 2654	2096	59
31	9.701 1839	2096	713 7935	2082	716 4017	2068	719 0085	2055	721 6140	2041	724 2181	2028	9.726 8210	2014	58
32	9.724 2210	2015	726 8223	2002	729 4227	1989	732 0217	1977	734 6193	1964	737 2157	1951	9.739 8109	1939	57
33	9.736 1109	1939	738 0048	1927	740 8972	1915	743 4883	1903	746 0792	1891	748 6683	1879	9.751 2562	1867	56
34	9.747 2562	1867	749 8427	1855	752 4283	1844	755 0128	1832	757 5960	1821	760 1782	1810	9.762 7591	1798	55°
35	9.758 3378	1797	760 8960	1788	763 4533	1778	766 0096	1768	768 5648	1757	771 1185	1747	9.773 6713	1737	54
36	9.769 4219	1733	771 9952	1723	774 4673	1712	776 9388	1702	779 4090	1692	781 9781	1682	9.784 5463	1672	53
37	9.779 4952	1671	781 1304	1661	784 1648	1651	786 6987	1641	789 2318	1632	791 7639	1622	9.794 2942	1612	52
38	9.789 2242	1612	791 8234	1602	794 2217	1593	796 8190	1583	799 4153	1574	801 9107	1565	9.804 4052	1555	51
39	9.798 8876	1555	800 9477	1546	803 4071	1537	806 0657	1528	808 5234	1519	810 9797	1510	9.813 4351	1502	50°
40	9.808 3907	1498	817 2967	1492	815 1961	1484	817 8946	1476	820 4919	1468	823 0885	1458	9.825 6843	1451	49
41	9.817 6793	1449	818 3972	1441	817 8332	1432	821 2685	1424	822 8032	1415	824 3364	1407	9.825 5211	1400	48
42	9.826 9211	1397	826 9110	1391	826 8301	1383	829 4483	1375	831 9658	1367	834 4825	1359	9.836 9983	1353	47
43	9.833 7833	1351	835 1184	1343	836 4477	1339	837 8762	1327	839 4040	1320	840 9309	1312	9.842 4571	1306	46
44	9.843 7771	1304	845 3178	1297	845 8278	1289	847 9366	1284	849 7448	1274	852 2518	1267	9.852 7585	1261	45°
45	9.853 4405	1260	850 7665	1252	853 7917	1252	855 4162	1238	857 9400	1231	860 4631	1224	9.862 9924	1218	44
46	9.856 9134	1216	858 4351	1209	859 9560	1202	860 5762	1195	861 7958	1188	862 7148	1181	9.864 1281	1175	43
47	9.866 1427	1175	865 3602	1168	866 4770	1161	867 6931	1154	868 7085	1148	869 9231	1141	9.871 0373	1135	42
48	9.871 1473	1134	872 4608	1128	873 5735	1121	874 6856	1114	875 7971	1108	876 9078	1101	9.877 7180	1095	41
49	9.877 7180	1092	878 8272	1088	879 7360	1082	880 8442	1076	882 1521	1070	883 2591	1063	9.884 4654	1059	40°
50	9.884 4654	1057	885 5711	1051	886 7762	1042	887 8806	1038	888 9844	1034	890 0877	1026	9.891 0904	1022	39
51	9.891 5203	1024	892 5243	1016	893 5278	1008	894 5304	1002	895 5325	996	896 5342	990	9.897 5352	986	38
52	9.897 9532	984	898 5518	978	899 5494	972	900 5467	966	901 5433	961	902 5394	955	9.903 5349	949	37
53	9.903 5349	949	904 5298	943	905 5241	938	906 5179	932	907 5111	926	908 5037	921	9.909 4958	915	36
54	9.909 4958	913	910 4873	909	911 4784	904	912 4688	898	913 4584	893	914 4477	887	9.915 4365	883	35°
55	9.915 4365	882	916 4240	876	917 4123	871	918 3994	867	919 3859	860	920 3719	855	9.921 3574	851	34
56	9.921 3574	847	919 4424	844	920 4268	839	921 4107	834	922 3940	828	923 3768	823	9.924 3591	819	33
57	9.924 3591	818	924 4409	813	925 4222	807	926 4029	802	927 4831	797	928 4629	792	9.929 4420	788	32
58	9.929 4420	787	929 4207	782	929 9989	777	930 9766	772	931 9537	767	932 8304	762	9.933 9066	758	31
59	9.933 9066	757	933 8822	752	934 8574	747	935 8320	746	936 8062	737	937 7799	732	9.938 7531	730	30°
60	9.938 7531	727	939 6258	722	940 6980	717	941 6697	716	942 6409	707	943 6117	703	9.944 5819	699	29
61	9.944 5819	698	945 5517	693	946 5210	688	947 4899	684	948 4582	679	949 4261	674	9.950 3935	670	28
62	9.950 3935	669	951 3604	665	952 3269	660	953 2929	655	954 2584	651	955 2235	646	9.956 1881	642	27
63	9.956 1881	641	957 0522	637	957 9159	632	958 7791	628	959 6419	623	960 5042	618	9.961 3660	614	26
64	9.961 3660	613	962 2274	607	963 0883	603	963 9488	599	964 8087	596	965 6681	591	9.966 5271	587	25°
65	9.966 5271	587	967 3856	582	968 2435	578	969 1009	574	969 9578	569	970 8142	565	9.971 6702	561	24
66	9.971 6702	560	972 5257	556	973 3808	551	974 2354	547	975 0895	543	975 9432	538	9.976 7964	535	23
67	9.976 7964	534	977 6490	530	978 5012	525	979 3529	521	980 2042	517	981 0550	513	9.981 9054	509	22
68	9.981 9054	508	982 7554	504	983 6048	500	984 4538	496	985 3023	492	986 1504	487	9.986 9980	484	21
69	9.986 9980	482	987 8452	478	988 6919	474	989 5382	470	990 3840	466	991 2294	462	9.992 0743	459	20°
70	9.992 0743	458	992 9188	454	993 7627	449	994 6062	446	995 4492	442	996 2918	437	9.997 1339	434	19
71	9.997 1339	433	997 9756	429	998 8168	425	999 6577	421	000 4977	417	001 3377	413	9.002 1768	410	18
72	9.002 1768	408	002 0155	404	002 8537	400	003 6916	396	004 5290	392	005 3660	388	9.006 2026	386	17
73	9.006 2026	384	006 9381	380	007 7731	376	008 6077	374	009 4419	368	010 2754	364	9.011 1086	362	16
74	9.011 1086	360	011 9412	356	012 7734	352	013 6051	350	014 4364	346	015 2673	342	9.016 0978	340	15°
75	9.016 0978	337	016 9280	333	017 7578	329	018 5872	327	019 4162	324	020 2447	321	9.021 0728	317	14
74	9.021 0728	313	021 9217	309	022 7526	305	023 5832	301	024 4133	297	025 2430	294	9.026 0724	290	13
77	9.026 0724	290	026 9014	286	027 7300	282	028 5582	278	029 3860	276	030 2134	270	9.031 0404	268	12
78	9.031 0404	267	031 8671	263	032 6934	259	033 5193	255	034 3448	251	035 1699	247	9.035 9947	245	11
79	9.035 9947	244	036 8190	240	037 6430	236	038 4666	232	039 2898	228	040 1127	225	9.040 9351	223	10°
80	9.040 9351	221	041 7572	217	042 5789	213	043 4003	210	044 2212	208	045 0418	204	9.045 8620	202	9
81	9.045 8620	198	046 6818	194	047 5013	191	048 3203	187	049 1390	183	049 9573	179	9.050 7753	177	8
82	9.050 7753	176	051 5928	172	052 4100	168	053 2269	164	054 0433	161	054 8594	157	9.055 6751	154	7
83	9.055 6751	153	056 4904	149	057 3053	146	058 1199	142	058 9341	138	059 7480	135	9.060 5614	134	6
84	9.060 5614	131	061 3743	127	062 1868	124	062 9989	120	063 8106	118	064 6220	113	9.065 4330	112	5°
85	9.065 4330	107	066 2436	103	067 0538	101	067 8637	98	068 6732	94	069 4824	90	9.070 2911	89	4
86	9.070 2911	87	071 0995	82	071 9076	79	072 7154	75	073 5229	72	074 3301	68	9.075 1369	67	3
87	9.075 1369	64	075 9433	61	076 7494	57	077 5552	53	078 3606	50	079 1657	46	9.079 9705	45	2
88	9.079 9705	42	079 7750	39	080 5791	35	081 3829	31	082 1864	28	082 9896	24	9.083 7924	23	1
89	9.083 7924	20	084 5949	17	085 3971	13	086 1989	9	087 0003	5	087 8014	2	10		0°
	60′		50′		40′		30′		20′		10′		0′		B

10 PLUS LOG COSINE B. WITH DIFFERENCES.

10 PLUS LOG TANGENT A, WITH R FUNCTION.

A	0′	10′	20′	30′	40′	50′	60′	
0°	USE T = 6.46375 OR, FOR GREATER ACCURACY, SEE SPECIAL TABLE OF S & T							89
1	8.241 421	8.300 884	8.360 093	8.418 068	8.463 849	8.505 267	8.543 084	88
	59.5 95	59.0 95	58.6 94	58.2 93	57.8 92	57.4 90	57.0 88	
2	8.300 084	8.359 877	8.419 487	8.477 968	8.534 160	8.589 529	8.643 119	87
	59.0 95	58.6 94	58.2 93	57.8 92	57.4 91	57.0 89	56.7 88	
3	8.358 179	8.418 422	8.478 265	8.536 886	8.593 451	8.648 826	8.703 044	86
	58.6 95	58.2 94	57.8 93	57.4 92	57.0 91	56.7 89	56.3 87	
4	8.416 944	8.476 516	8.536 543	8.595 278	8.651 866	8.707 315	8.761 926	85°
	58.2 95	57.8 94	57.4 93	57.0 92	56.7 91	56.3 89	55.9 88	
5°	8.475 139	8.534 781	8.594 170	8.653 317	8.710 282	8.765 804	8.820 620	84
	57.8 95	57.4 94	57.0 93	56.7 92	56.3 91	55.9 90	55.5 88	
6	8.533 234	8.593 045	8.652 804	8.712 511	8.769 173	8.824 694	8.879 464	83
	57.4 95	57.0 94	56.7 93	56.3 92	55.9 91	55.5 90	55.1 89	
7	8.591 329	8.651 309	8.711 439	8.771 429	8.829 067	8.884 584	8.939 147	82
	57.0 95	56.7 94	56.3 93	55.9 92	55.5 91	55.1 90	54.7 89	
8	8.649 424	8.709 577	8.769 774	8.829 544	8.888 959	8.944 474	8.999 713	81
	56.7 95	56.3 94	55.9 93	55.5 92	55.1 91	54.7 90	54.3 89	
9	8.707 519	8.767 845	8.828 115	8.888 036	8.948 050	9.003 364	9.058 902	80°
	56.3 95	55.9 93	55.5 92	55.1 91	54.7 91	54.3 90	53.9 89	
10°	8.765 614	8.826 113	8.886 456	8.946 727	9.006 991	9.062 054	9.118 052	79
	55.9 95	55.5 94	55.1 93	54.7 92	54.3 91	53.9 90	53.5 89	
11	8.823 709	8.884 383	8.945 051	9.005 653	9.065 982	9.121 444	9.176 941	78
	55.5 95	55.1 94	54.7 93	54.3 92	53.9 91	53.5 90	53.1 89	
12	8.881 804	8.942 653	9.003 646	9.064 579	9.124 973	9.180 834	9.236 830	77
	55.1 95	54.7 94	54.3 93	53.9 92	53.5 91	53.1 90	52.7 89	
13	8.939 899	9.000 923	9.062 241	9.123 505	9.183 964	9.241 224	9.296 771	76
	54.7 95	54.3 94	53.9 93	53.5 92	53.1 91	52.7 90	52.3 89	
14	8.998 094	9.059 194	9.120 837	9.182 431	9.242 955	9.301 614	9.356 771	75°
	54.3 95	53.9 94	53.5 93	53.1 92	52.7 91	52.3 90	51.9 89	
15°	9.056 389	9.117 465	9.179 357	9.241 358	9.301 947	9.360 004	9.416 771	74
	53.9 95	53.5 94	53.1 93	52.7 92	52.3 91	51.9 90	51.5 89	
16	9.114 684	9.175 736	9.237 877	9.299 285	9.360 938	9.418 394	9.475 771	73
	53.5 95	53.1 94	52.7 93	52.3 92	51.9 91	51.5 90	51.1 89	
17	9.172 979	9.234 007	9.296 398	9.358 138	9.418 929	9.476 784	9.534 771	72
	53.1 95	52.7 94	52.3 93	51.9 92	51.5 91	51.1 90	50.7 89	
18	9.231 274	9.292 278	9.354 919	9.417 091	9.478 920	9.537 174	9.595 772	71
	52.7 95	52.3 94	51.9 93	51.5 92	51.1 91	50.7 90	50.3 89	
19	9.289 569	9.350 549	9.413 440	9.476 044	9.537 911	9.597 564	9.656 772	70°
	52.3 95	51.9 94	51.5 93	51.1 92	50.7 91	50.3 90	49.9 89	
20°	9.347 864	9.408 820	9.471 961	9.534 996	9.596 902	9.655 954	9.714 772	69
	51.9 95	51.5 94	51.1 93	50.7 92	50.3 91	49.9 90	49.5 89	
21	9.406 159	9.467 091	9.530 482	9.593 948	9.655 893	9.715 344	9.774 772	68
	51.5 95	51.1 94	50.7 93	50.3 92	49.9 91	49.5 90	49.1 89	
22	9.464 454	9.525 362	9.589 003	9.652 900	9.714 884	9.774 734	9.834 772	67
	51.1 95	50.7 94	50.3 93	49.9 92	49.5 91	49.1 90	48.7 89	
23	9.522 749	9.583 633	9.647 524	9.711 852	9.773 875	9.834 124	9.894 772	66
	50.7 95	50.3 94	49.9 93	49.5 92	49.1 91	48.7 90	48.3 89	
24	9.581 044	9.641 904	9.706 045	9.770 804	9.832 867	9.893 514	9.954 772	65°
	50.3 95	49.9 94	49.5 93	49.1 92	48.7 91	48.3 90	47.9 89	
25°	9.639 339	9.700 175	9.764 566	9.829 757	9.891 860	9.952 904	10.014 772	64
	49.9 95	49.5 94	49.1 93	48.7 92	48.3 91	47.9 90	47.5 89	
26	9.697 634	9.758 446	9.823 088	9.888 711	9.950 854	10.012 294	10.074 772	63
	49.5 95	49.1 94	48.7 93	48.3 92	47.9 91	47.5 90	47.1 89	
27	9.755 929	9.816 717	9.881 611	9.947 666	10.009 849	10.071 684	10.134 772	62
	49.1 95	48.7 94	48.3 93	47.9 92	47.5 91	47.1 90	46.7 89	
28	9.814 224	9.874 988	9.940 135	10.006 622	10.068 845	10.131 074	10.194 772	61
	48.7 95	48.3 94	47.9 93	47.5 92	47.1 91	46.7 90	46.3 89	
29	9.872 519	9.933 259	9.998 659	10.065 579	10.127 841	10.190 464	10.254 772	60°
	48.3 95	47.9 94	47.5 93	47.1 92	46.7 91	46.3 90	45.9 89	
	60′	50′	40′	30′	20′	10′	0′	B

10 PLUS LOG COTANGENT B, WITH R FUNCTION.

10 PLUS LOG TANGENT A WITH DIFFERENCES.

A	0′		10′		20′		30′		40′		50′		60′	
30°	9.761 439	2913	.764 352	2903	.767 255	2893	.770 148	2885	.773 033	2875	.775 908	2866	.778 774	59
31	9.778 774	2857	.781 631	2848	.784 479	2840	.787 319	2832	.790 151	2823	.792 974	2815	.795 789	58
32	9.795 789	2807	.798 596	2799	.801 396	2792	.804 187	2784	.806 971	2777	.809 748	2769	.812 517	57
33	9.812 517	2762	.815 280	2755	.818 035	2748	.820 783	2741	.823 524	2735	.826 259	2728	.828 987	56
34	9.828 987	2722	.831 709	2716	.834 425	2709	.837 134	2703	.839 838	2697	.842 535	2692	.845 227	55
35°	9.845 227	2686	.847 913	2680	.850 593	2675	.853 268	2670	.855 938	2664	.858 602	2659	.861 261	54
36	9.861 261	2654	.863 915	2649	.866 564	2645	.869 209	2640	.871 849	2635	.874 484	2631	.877 114	53
37	9.877 114	2626	.879 741	2622	.882 363	2618	.884 980	2614	.887 594	2610	.890 204	2606	.892 810	52
38	9.892 810	2602	.895 412	2598	.898 010	2595	.900 605	2591	.903 197	2588	.905 785	2585	.908 369	51
39	9.908 369	2581	.910 951	2578	.913 529	2575	.916 104	2573	.918 677	2570	.921 247	2567	.923 814	50°
40°	9.923 814	2564	.926 378	2562	.928 940	2559	.931 499	2557	.934 056	2555	.936 611	2553	.939 163	49
41	9.939 163	2550	.941 713	2549	.944 262	2547	.946 808	2545	.949 353	2543	.951 896	2541	.954 437	48
42	9.954 437	2540	.956 977	2538	.959 516	2537	.962 052	2536	.964 588	2534	.967 123	2533	.969 656	47
43	9.969 656	2532	.972 188	2531	.974 720	2530	.977 250	2530	.979 780	2529	.982 309	2528	.984 837	46
44	9.984 837	2528	.987 365	2527	.989 893	2527	.992 420	2527	.994 947	2527	.997 473	2527	10	45°
	60′		50′		40′		30′		20′		10′		0′	B

10 PLUS LOG COTANGENT B, WITH DIFFERENCES.

S AND T FOR ANGLES EXPRESSED IN MINUTES.

	a =	0	1	2	3	4	5	6	7	8	9
A = 0.	S = 6.463	726	726	726	726	726	726	726	726	726	726
	T = 6.463	726	726	726	726	726	726	726	727	727	727
10		726	725	725	725	725	725	725	724	724	724
		727	727	728	728	728	729	729	729	730	730
20		723	723	723	723	722	722	722	721	721	721
		731	731	732	732	733	734	734	735	735	736
30		720	720	720	719	719	718	718	717	717	717
		737	738	738	739	740	741	742	743	744	744
40		716	716	715	715	714	713	713	712	712	711
		745	746	747	748	750	751	752	753	754	755
50		711	710	709	709	708	707	707	706	705	705
		756	758	759	760	762	763	764	766	767	769
60		704	703	702	702	701	700	699	698	697	697
		770	771	773	774	776	778	779	781	783	784
70		696	695	694	693	692	691	690	690	689	688
		786	788	789	791	793	795	797	799	800	802
80		687	686	685	684	683	682	681	680	678	677
		804	806	808	810	812	814	817	819	821	823
90		676	675	674	673	672	671	669	668	667	666
		825	827	830	832	834	836	839	841	844	846
100		665	663	662	661	660	658	657	656	654	653
		848	851	853	856	858	861	864	866	869	871
110		652	650	649	648	646	645	643	642	641	639
		874	877	880	882	885	888	891	895	896	899

2

The Present Theory of Switching and Some of its Future Trends

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Our lot should not be so much one of wanting to construct machines to do social-humanistic endeavors, but more one of wanting to keep man from doing any classical machine-like routines.

0. *Introduction*: Since early time it has been recognized that the transportation of a note by a messenger usually entailed a long duration of time before the recipient received the note. Such a situation placed nations with effective communication and transportation channels in a more favored political position than nations without such effective communication and transportation channels. The Appian Way and its connecting roads are a testament to the importance of communication and transportation in the maintenance for 400 years of a social-political-economic system called The Roman Empire. However, during the 19th century it became clear that the relative control of electricity, with its high velocity of propagation, offered a means of transporting messages to any place on earth in a matter of a fraction of a second. Such an idea soon led to the invention and development of telegraph, telephone, radio and television. In fact, this continuous emphasis on communication has led Wiener [1,2,3] and Shannon [4] to put forth the axiological proposition that the two most important functions of each human are (1) to attempt more adequately to control his environment and (2) to communicate with his fellow man.

The creation of the dial telephone system and its complex switching apparatus necessary to join parties together, coupled with the speed and reliability of calculations that could be offered by computing machinery led to the development of a systematic procedure for treating switching circuits. A switching circuit is an electric circuit composed only of components each having its electrical property described by being in either one or the other of two different states. This paper is an expository treatment of those mathematical aspects of switching theory that the author, at present, believes to be important.

1. *History*: The basic principles upon which the theory of switching circuits has been developed are those that treat the classical Aristotelian syllogism [5]. During the 18th century the Swiss mathematician Leonard Euler [6] conceived the idea of treating syllogisms by means of logical diagrams and a few simple rules of manipulation. By a "logical diagram" is meant a diagram composed of dots and curves in which logical relations are denoted by such spatial relations that the necessary consequences of these logical relations are placed in clear evidence by the rules of manipulation. Near the middle of the 19th century the English mathematicians George Boole [7,8] and Augustus DeMorgan [9] conceived an algebra of conventional logic. This made syllogistic reasoning amendable to the more conventional algebraic manipulations. Toward the end of the 19th century the American mathematician and philosopher C. S. Peirce [10,11] considerably extended and developed Boole's and DeMorgan's algebra of logic. In addition, it is quite fair to consider Peirce as the inventor of the now familiar truth-table. In the last decade of the 19th century there appeared a book by the English logician John Venn [12] in which he considerably refined Euler's idea of the use of logic diagrams. In fact, it was Venn who coined the term "Symbolic Logic." In 1910, the Russian mathematician P. Eréńfést [13], in his review of the translated copy of a book by the French mathematician L. Couturant [14] suggested the relation between two-valued logic and relay switching circuits. Supposedly, in 1935, the Russian mathematician V. I. Šeśtakov developed (but published in 1941) Eréńfést's proposal. Further amplification of his results appears in [15]. Between 1936 and 1938 there were numerous fundamental papers published which developed the pure and applied aspects of the theory of Boolean Algebra [16,17,18,19]. The Japanese engineers Nakasima and Hanzamus gave an application of Boolean algebra to switching circuits that preceeded the more thoroughly developed works on the same subject by C. E. Shannon [20,21]. About the middle of World War 2, the classical paper by W. S. McCulloch and W. Pitts [22] was written which described an application of logic to a better understanding of the switching properties associated with the functioning of the human brain. In 1949, Shannon [23] published many new results concerning spring load distributions, contact load distributions and partially symmetric functions none of which are considered in [20,21]. In 1955, F. E. Hohn and L. R. Schissler [24] published a paper on the use of matrices in the study of switching circuits. However, some of their results were already anticipated in [20]. In 1956, John von Neumann [25] published the results of his 1952 lectures on Automata Theory that he gave at the California Institute of Technology. Certain parts of his paper are an extension and ramification of the results of [22]. In addition, he showed that

under certain conditions unreliable components could be coupled together so as to give arbitrarily reliable systems. At about this time D. A. Huffman [26], G. H. Mealy [27] and E. F. Moore [28] published their models for introducing the parameter of time into switching circuits. Some papers specifically directed at minimizing the number of components needed in the realization of Boolean functions are those of W. v.O. Quine [29,30], E. J. McCluskey, Jr. [31] and S. R. Petrick [32]. Moore and Shannon [33] consider the intuitive ideas presented in [25] and apply them to relay contact networks. In addition, they develop an interesting class of error-reducing network configurations called 'hammock networks.' D. E. Muller [34] applies the lattice theory of Garrett Birkhoff [35] to the analysis and synthesis of asynchronous switching circuits. A. A. Mullin extends the results of [33] to considerations with a parameter of time [36] and space [37]. Then he introduces the concept of a switching field [38] as a generalization of a switching circuit and metric reliability [39] as a unifying class of reliabilities. W. G. Kellner and he then treat a necessary condition for a minimum component of Boolean functions [40]. C. Y. Lee [41] considers a technique for representing switching circuits that is more in the form of a computer program. This procedure has the obvious side-advantage of giving a convenient means of specifying a measure of complexity for a Boolean function. At present, there are five mathematics or engineering textbooks of a general nature on the subject of logical design [42,43,44,45,59].

2. *Classical Theory*:

2.1. *Preliminaries*: Section 2 of the paper treats the classical aspects of switching circuits. The word "classical" is used in the sense of involving the deterministic procedures of Boolean algebra for the study of switching circuits as proposed by Shannon [20,21,23]. In the paper we adopt the transmission function conventions [42]. Thus, '0' denotes an open circuit, '1' denotes a closed circuit, '+' denotes parallel connections of contacts and '·' denotes series connection of contacts. The classical theory of switching circuits is treated in two parts: 2.2. Classical combinational switching circuits and 2.3. classical sequential switching circuits. For brevity, when in text, we drop the qualifying adjective 'classical' in referring to 'classical combinational' and 'classical sequential' switching circuits. 'Classical' is to be assumed present in remarks and only remarks concerning the 'classical theory.' By a combinational switching circuit is meant a switching circuit whose electrical property is described, by means of Boolean algebra, only in terms of a sequence consisting of the states of each of the input variables. For example, a network composed exclusively of relay contacts is a

combinational circuit. By a sequential circuit is meant a switching circuit whose electrical property is described by means of lattice theory only in terms of the sequence consisting of the states of each of the input variables including the order in which each member of the set of input variables arrived at its state; for example, a network in which a contact for a relay also controls the excitation to the coil of the relay.

2.2. *Classical Combinational Switching Circuits*:

2.2.1. *Preliminaries*: The theory of section 2.2 is treated in two parts: 2.2.2 minimization theory and 2.2.3 unilateral iterative arrays. For practical reasons the treatment of combinational circuits will not include contact-load and spring-load distribution problems [23], partially symmetric Boolean functions [31], bilateral iterative arrays, general multi-terminal contact networks [42], nor the general synthesis of non-series-parallel contact networks [24,31].

2.2.2. *Minimization Theory*:

2.2.2.1. *Preliminaries*: By minimization theory we mean a procedure aimed at reducing the number of entities in a circuit that have a certain characterization. We consider only the problem of reducing the number of relay contacts or the number of electron tubes needed to realize a given Boolean Function.

The theory of section 2.2.2 is treated in four parts: 2.2.2.2. algebraic simplification, 2.2.2.3. symmetric functions, 2.2.2.4. a residue test, and 2.2.2.5. the Quine-McCluskey reduction.

2.2.2.2. *Algebraic Simplification*: Since each literal [29] in a Boolean function is to be realized by a component, one is motivated to apply identities for reducing the number of literals in a representation for a given Boolean function. Eight such identities are the following:

(1) $x + xy = x$
(2) $x(x+y) = x$
(3) $(x+y')y = xy$
(4) $xy'+y = x+y$
(5) $xy+xz = x(y+z)$
(6) $(x+y)(x+z) = x + yz$
(7) $xy+yz+zx' = xy+zx'$
(8) $(x+y)(y+z)(z+x') = (x+y)(z+x')$

Consider the circuit shown in Figure 1. Let T_1 denote the transmission function describing the electrical properties of the circuit shown. Then

$T_1 = A(C + BD) + (BC + B')(A' + D)$, or by (4)

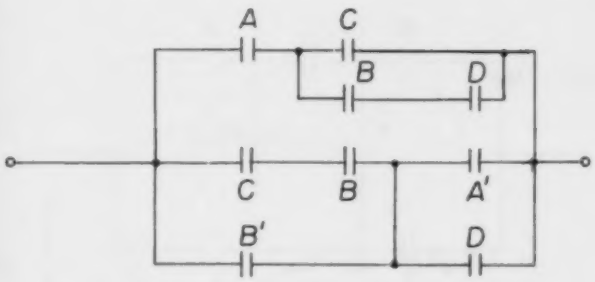

Figure 1. A Combinational Circuit (9 contacts)

$= A(C + BD) + (B' + C)(A' + D)$, or by (1),(4) and (7)

$= C + AD + A'B'$.

The simplified circuit for T_1 is shown in Figure 2. Thus, four contacts have been saved by some simple algebraic manipulations.

If the original two-terminal network is non-series-parallel, the transmission function is found by tracing all paths between the two terminals. Consider the circuit shown in Figure 3. Then

$$T_2 = wzx + wzw'y + wy'x'y + zx'y + zx'w'x + zy'zx$$

or $$T_2 = z(x'y + x(w + y')).$$

The simplified circuit for T_2 is shown in Figure 4. Thus, two contacts are saved by the algebraic manipulation.

2.2.2.3. *Symmetric Functions*: An economical class of switching circuits is the one that realizes the class of symmetric functions. The basic concept of a symmetric function S of n variables is that T = 1 if and only if some specified number of the variables are 1. For example, $S_{2,3}(x_3, x_2, x_1) = x_1x_2 + x_2x_3 + x_3x_1$ is a

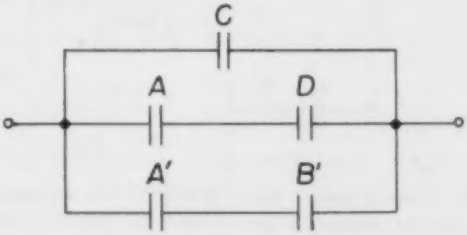

Figure 2. Simplified Circuit for T_1 (5 contacts)

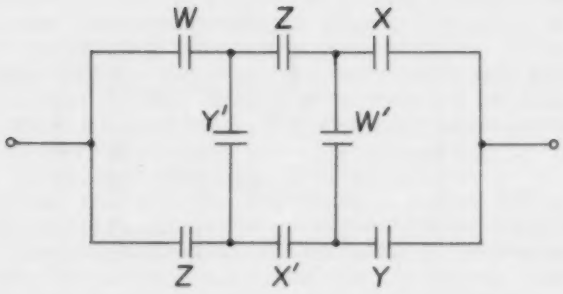

Figure 3. A Non-Series-Parallel Circuit

symmetric function that takes on the value 1 if and only if at least two variables (i.e. either 2 or 3 variables) take on the value 1. Figure 5 shows the basic circuit for generating all of the symmetric functions on three variables. By an iterative process the structure is easily extended to consider the case of any given number of variables.

2.2.2.4. *A Residue Test*: A residue test [40] is a convenient means for determining the essential literals [29] in the

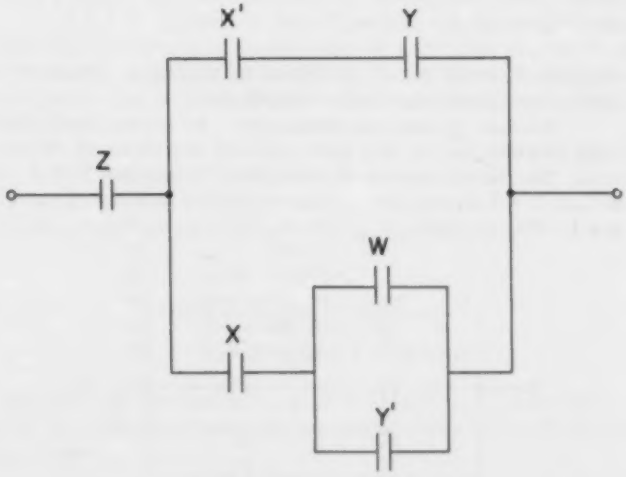

Figure 4. Simplified Circuit for T_2 (6 contacts)

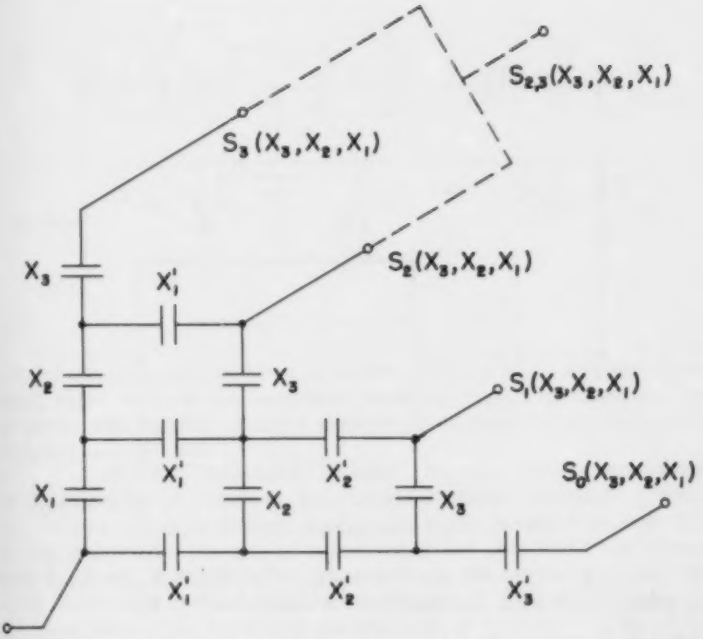

Figure 5. Generating all symmetric functions on three variables

transmission function describing the electrical properties of a two-terminal combinational switching circuit [23], and thus indicates which literals are absolutely necessary in a realization of the switching circuit.

The residue test is based upon a pair of identities of the Boolean algebra related by duality. These identities are 1 and 1′ below, where T is a transmission function.

$$(1)\quad T = x_k T_{1k} + x_k' T_{0k}$$

$$(1')\quad T = [x_k + T_{0k}] \cdot [x_k' + T_{1k}],$$

where

$$T = T(x_n, \ldots, x_{k+1}, x_k, x_{k-1}, \ldots, x_0)$$

$$T_{1k} = T(x_n, \ldots, x_{k+1}, 1, x_{k-1}, \ldots, x_0)$$

$$T_{0k} = T(x_n, \ldots, x_{k+1}, 0, x_{k-1}, \ldots, x_0)$$

Consider the following three cases, in which the inclusion relation is the relation of possessing the same terms of standard sum (in its canonic expansion) [42]:

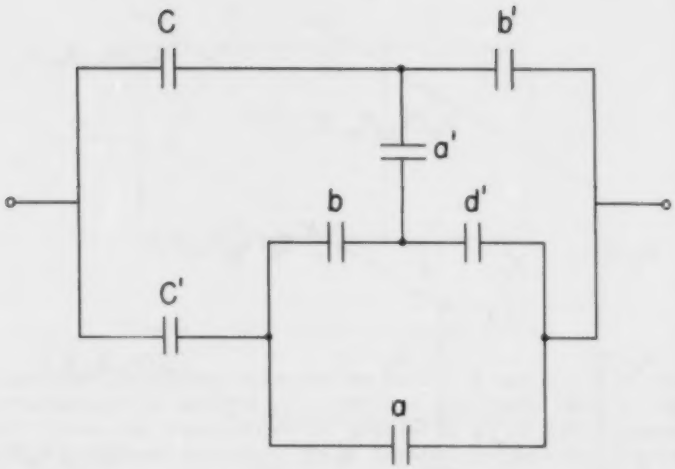

Figure 6. A minimum contact circuit

Case 1: $T_{1k} = T_{0k}$. Then $T = x_k T_{1k} + x'_k T_{0k} = T_{0k} = T_{1k}$. Here neither x_k nor x'_k is necessary for a realization of T.

Case 2:

(a) $T_{1k} \supset T_{0k}$ but $T_{1k} \neq T_{0k}$ (i.e. $T_{1k} \neq T_{0k}$, and if $T_{0k} = 1$ then $T_{1k} = 1$). Now $T_{0k} \cdot T_{1k} = T_{0k}$ and $T_{0k} + T_{1k} = T_{1k}$. Then $T = x_k(T_{0k} + T_{1k}) + x'_k T_{0k} = x_k T_{0k} + T_{0k}$. Here x'_k is not necessary for a realization of T.

(b) $T_{0k} \supset T_{1k}$ but $T_{1k} \neq T_{0k}$ (i.e. $T_{1k} \neq T_{0k}$ and if $T_{1k} = 1$ then $T_{0k} = 1$). By reasoning similar to case (2a), the literal x_k is not necessary in a realization of T.

Case 3: None of the above. Here no reduction in the number of literals is possible.

If a circuit is found that uses each necessary literal (found according to the residue test) only once then the circuit is necessarily a minimum component circuit. Consider the circuit shown in Figure 6. The residue test shows that the only necessary literals are a, a′, b, b′, c, c′, d′. Since each of these occurs at most once the circuit shown in Figure 6 is a minimum contact circuit.

2.2.2.5. *Quine-McCluskey Reduction*: The Quine-McCluskey [29,30,31] reduction procedure is essentially a systematic procedure for effecting all of the reduction identities of section

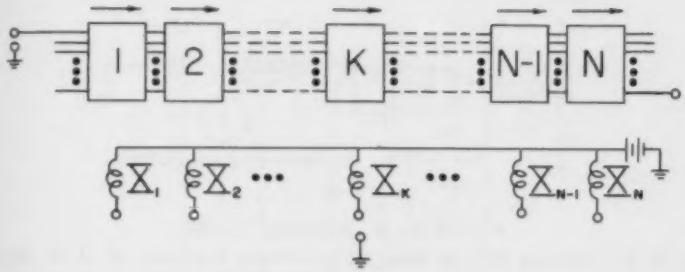

Figure 7. An Iterative Array

2.2.2.2. Such a scheme is most useful to the circuit designer. However, since no new mathematical ideas are concerned we will not consider the details. Such a detailed treatment is carried out in Caldwell's book [42].

2.2.3.1. *Unilateral Iterative Arrays*: An iterative array is a collection of identical 2m-terminal contact networks coupled together as shown in Figure 7. Such an array is called unilateral to the right provided the circuit structure of the identical cells is such that a ground placed at the left of the entire array can propagate from each cell only to cells on its right. A typical transmission function that is economically realized by an iterative array is the following: T = 1 if and only if there is a single consecutive group of three operated relays. A typical cell for such a problem is shown in Figure 8.

2.3. *Classical Sequential Switching Circuits*:

2.3.1. *Preliminaries*: The theory of section 2.3 is treated in one part: 2.3.2 minimization theory. For practical reasons the treatment of sequential circuits will not include the recently developed work by E. F. Moore [28] on sequential machines, the work

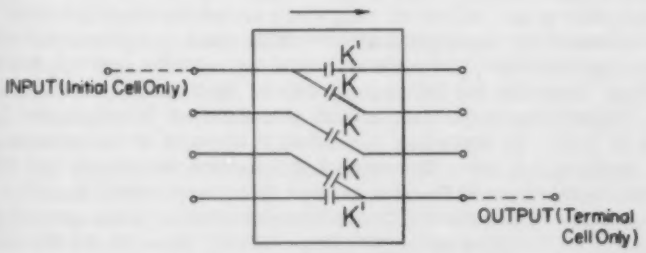

Figure 8. A Typical Cell

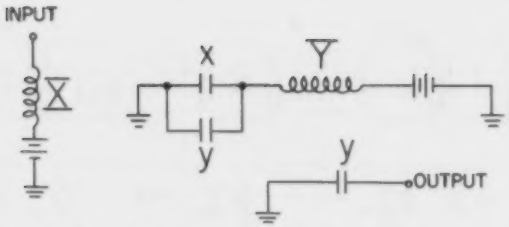

Figure 9. A "Lock-up" Circuit

by D. A. Huffman [47] on delay polynomials, the work of C. C. Elgot [48,49] on automata theory, the work of W. G. Kellner [50] on iterative sequential circuits, pulsed sequential circuits [42], the recent work of Seiiti Huzino [51,52,53,54] on sequential machines, nor the work of J. P. Roth [55] on the application of algebraic topology to the synthesis of switching circuits.

2.3.2. *Minimization Theory*:

2.3.2.1. *Preliminaries*: We use minimization theory in the same sense as in section 2.2.2. The theory of section 2.3.2. is treated in one section 2.3.2.2. the Huffman model. For a minimization theory for other models see Moore [28] and Mealy [27]. We use the convention that a Boolean variable associated with a relay-coil is to be an upper-case letter and the Boolean variable associated with contacts on that relay is to be denoted by the same lower-case letter.

2.3.2.2. *The Huffman Model*: The Huffman model for a sequential circuit [26] consists of a number of combinational circuits some of which control the excitation to relay coils (or other memory devices). However, some of the combinational circuits receive a kind of 'feedback' from these coils in the sense that the coils actuate contacts in the combinational circuits. A simple example of a sequential circuit is the 'lock-up' circuit shown in Figure 9. The relays under our direct control at all time (input) are called 'primary relays' and are denoted by an upper-case 'x' with the appropriate subscript. All other relays are called 'secondary relays' and are denoted by an upper-case 'y' with the appropriate subscript. The output is denoted by a lower-case 'z' with the appropriate subscript. Consider the following problem: Synthesize a two-input and one output relay sequential circuit whose output transmission function is in the one state (i.e. grounded) if and only if the ordered-pair of inputs x_1, x_2 form the connected sequence 00, 10, 11 and whose output transmission function is zero (i.e. ungrounded) for all other input-pair combinations. Double changes of input state are not permitted (i.e. the input transition 00 ⟷ 11 and 01 ⟷ 10 are forbidden) in order to prevent 'race' conditions.

$X_1 X_2$ 00	01	11	10	Z
(1)	4	—	2	0
1	—	3	(2)	0
—	4	(3)	6	1
1	(4)	5	—	0
—	4	(5)	6	0
1	—	5	(6)	0

Figure 10. A Primitive Flow Table

The classical synthesis procedure consists of the following:

(1) Construct a *primitive flow table* with one stable-state per row. We know that at least one secondary relay must be used since there are only four possible input states. We denote stable-states by circled entries. An uncircled state seeks its correspondingly numbered circled state. Forbidden changes from a stable-state are denoted by a dash. Each stable-state determines a unique output-state and each stable-state depends only upon the previous stable state and the present input state x_1, x_2. Figure 10 shows a primitive flow table for the problem.

(2) Next check the primitive flow table for inaccessible or equivalent states. Equivalent states at most differ in their notation in the sense that they lead to the same states for all input transitions. No states of the primitive flow table shown in Figure 10 are inaccessible nor is any pair of states equivalent.

(3) In an effort to reduce the number of rows in the primitive flow table and hence possibly the number of secondary relays, a merger diagram is constructed. Such a diagram places in evidence all possible row mergers of the primitive flow table. Two or more rows can be merged, if column-wise, they have the same numerics

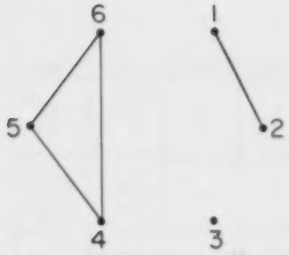

Figure 11. A Merger Diagram

(disregarding whether an entry is circled or not) representing states. A node of the merger diagram corresponds to a row of the primitive flow table. The numbering of the nodes of the diagram corresponds to the stable state of the row of the primitive flow table. Join a straight line between each pair of nodes of the merger diagram that corresponds to rows that can be merged. A merger diagram for the problem is shown in Figure 11.

(4) The merged flow table formed by making only the 4,5,6 merger is shown in Figure 12. The dashed entries of the primitive flow table are chosen as the dotted numbers in the merged flow table indicate.

(5) Now make an assignment of secondary states to the rows of the merged flow table (labeled a,b,c,d as in Figure 12) by means of a transition diagram. A transition diagram shows all interrow

X_1X_2	00	01	11	10
a	①	4	5	2
b	1	4	3	②
c	1	4	③	6
d	1	④	⑤	⑥

Figure 12. A Merged Flow Table

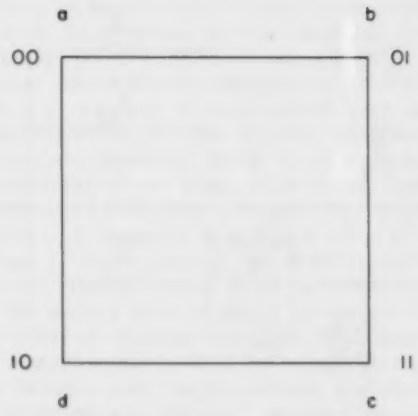

Figure 13. A Transition Diagram

transitions of the merged flow table by means of straight lines joining the nodes (which are identified with rows of the flow table) of the transition diagram. Figure 13 shows a transition diagram for the problem. Note that there is no difficulty with interrow transitions for the two left-hand columns of the merged flow table, since for these columns, there is no chance of going to any state but the desired state.

$y_1 y_2$ \ $x_1 x_2$	00	01	11	10
00	00	10	10	01
01	00	10	11	01
11	00	10	11	10
10	00	10	10	10

$Y_1 Y_2$

EXCITATION MATRIX

$y_1 y_2$ \ $x_1 x_2$	00	01	11	10
00	0	0	0	0
01	0	0	1	0
11	0	0	1	0
10	0	0	0	0

Z

OUTPUT MATRIX

Figure 14. An Excitation and Output Matrix

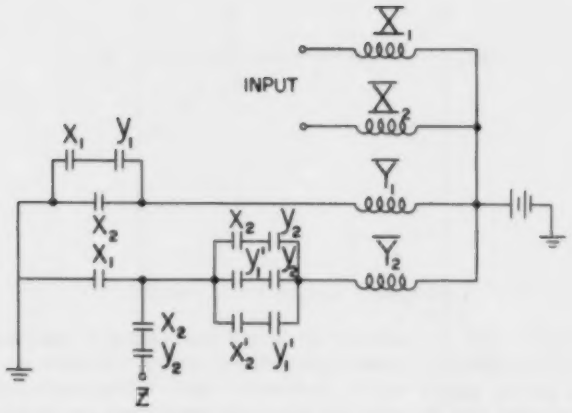

Figure 15. A Circuit for the Problem

(6) Next write a secondary coil-excitation matrix and an output matrix each corresponding to the choice of secondary assignment to the merged flow table. A transmission function of 1 is associated only with state 3 of the merged flow table (this is a result of having been built into the primitive flow table). An excitation matrix and an output matrix are shown in Figure 14.

Hence,

$$Y_1 = x_2 + x_1 y_1$$

$$Y_2 = x_1 (x_2 y_2 + x'_2 y'_1 + \underline{y'_1 y_2})$$

$$z = (x_1 x_2 y_2).$$

The underlined term in Y_2 is to remove the effect of a static hazard [46]. Otherwise, if the circuit was in stable-state 2 and we changed the input to $x_1, x_2 = 1,1$, then the excitation to coil Y_2 could possibly be turned off momentarily. In general, the rule [46] is to include any prime implicants [29], necessary to prevent either a 1-0-1 or 0-1-0 sequence of a transmission function controlling secondary relay coils.

(7) A circuit for the problem is shown in Figure 15. Note that we can use the minimization techniques of combinational circuits to get minimal circuits for Y_1, Y_2 and z.

3. *Stochastic Theory*:

3.1. *Preliminaries*: Section 3 of the paper treats the statistical aspects of the theory of switching. The word 'stochastic' is used in

the sense of involving the statistical procedures concerning fallible switching elements as proposed by von Neumann [25], Moore and Shannon [33], Mullin [36,37] and more recently [56].

The stochastic theory of switching may be divided into two parts. The first part is a calculus of probabilities built upon the superstructure of the classical Boolean algebra according to the usual postulates for probability theory, put forth by A. N. Kolmogorov [57]. Thus, if all probabilities are either zero or one then we return to the classical theory of switching circuits. The second part of the stochastic theory is a measure theory in a space of Boolean functions. These two problems are inherently different in nature. The first is suited to errors resulting in the m-triplication of Boolean variables and the second part is suited to errors resulting in the composition of different Boolean variables. The first part is applicable to relay contact networks and the second part is applicable to potential-level devices like electron-tubes, transistors and cryotrons. In this paper we will consider only the first part dealing with a calculus of probabilities and called the 'relay case.' We will not treat the second part, but refer the reader to the works of von Neumann [25] and McCulloch [58]. The 'relay case' is divided into two parts: 3.2. relay stochastic combinational switching and 3.3. relay stochastic sequential switching. For brevity, when in text, we drop the qualifying adjectives 'relay stochastic' in referring to 'relay stochastic combination' and 'relay stochastic sequential' switching. 'Relay stochastic' is to be assumed present in remarks and only remarks concerning the 'stochastic theory.'

3.2. *Relay Stochastic Combinational Switching*:

3.2.1. *Preliminaries*: The theory of section 3.2. is treated in four parts: 3.2.2. the model and reliability, 3.2.3. error-reducing configurations, 3.2.4. iteration processes, and 3.2.5. switching fields. The adjective 'combinational' is used in the same sense as for the classical case as described in section 2.1.

3.2.2. *The Model and Reliability*: In the model for stochastic circuits each relay contact is defined by some binary information channel as shown in Figure 16. Here 'a' is the probability that the relay contact is open if the relay coil is deenergized. However, for simplicity, we assume that a+b = 1, that each and every relay contact is statistically specified by the same defining channel and that contacts function and malfunction statistically independent of one another. To this extent the probability p_e of a contact malfunction is $p_e = 1-a$.

By definition the reliability ρ of a contact or a circuit of contacts is expressed in terms of its probability of malfunction p_e by $\rho = -\ln p_e$.

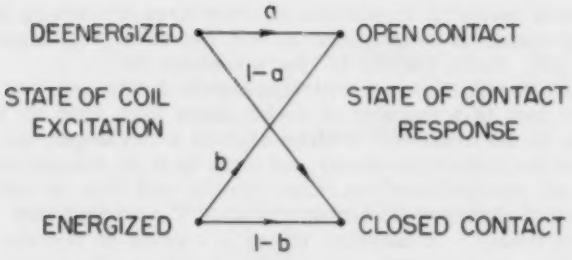

Figure 16. A Defining Channel for a Relay Contact

3.2.3. *Error-Reducing Configurations*: Let a and b be the parameters of a defining channel for contacts. When a relay contack is replaced by a two-terminal network of contacts on the same relay then the network of contacts has an equivalent defining channel as shown in Figure 17, where α is some probability function of a and b and β is some probability function of a and b (according to the conventions of section 3.2.2.). If $\beta < b$ and $\alpha < a$ then the network is said to be an error-reducing configuration. Figure 18 shows two elements from the class of all error-reducing configurations.

3.2.4. *Iteration Processes*: Let $T^{(1)}$ denote a specified combinational circuit topology. Let Z be the set of positive integers. Let $T^{(j+1)}$, $j \in Z$, be the topology of the circuit formed from the topology $T^{(j)}$, $j \in Z$, by replacing each of its contacts with a circuit of topology $T^{(1)}$. Under these conditions $T^{(\lambda)}$, $\lambda \in Z$ is said to be the λ-iterate of $T^{(1)}$. The process under consideration is said to be the topological iteration of $T^{(1)}$. It can be shown [33] that every

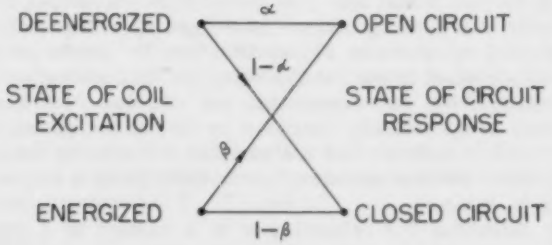

Figure 17. A Defining Channel for a Two-Terminal Contact Network

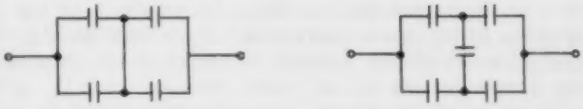

Figure 18. Two Error-Reducing Configurations

iterate of an element from the class of error-reducing configurations is also an element of the class of error-reducing configurations.

3.2.5. *Switching Fields*: Consider an isovolumetric topological iteration process [37]. By this is meant the following: (i) an iteration of network topology using some element of the (non-vacuous) class of error-reducing configurations, and (ii) each network component has a reliability function defined on some non-vacuous index set of real-valued parameters such that the product of the total number of components by each of these parameters is a constant.

By considering the circuit-reliability function, which is a real-valued function of network topology and component reliability, in the limit (if the limit exists) as each parameter of the domain of the circuit-reliability function approaches zero, one is led to an entity that may be called a homogeneous, isotropic combinational switching field.

3.3. *Relay Stochastic Sequential Switching*:

3.3.1. *Preliminaries*: The theory of section 3.3 is treated in two parts: 3.3.2. the model, and 3.3.3. stochastic matrices, error matrices and reliability matrices.

3.3.2. *The Model*: By analogy to the case of classical sequential circuits, the model for sequential circuits is to be a number of combinational circuits some of which control relay coils which have contacts in the combinational circuits. Then, by the techniques of section 3.2. we proceed to make each combinational circuit so reliable that the over-all sequential circuit has the desired reliability properties. We will not consider the case for which hazards are present. Such problems lead into the concept of a probabilistic prime implicant [36].

3.3.3. *Stochastic matrices, error matrices and reliability matrices*: A stochastic matrix is an exhaustive tabulation (analogous to the primitive flow table of classical theory) of each and every manner in which all the components function or malfunction, together with the transition probabilities between these states as determined by the defining channels for the combinational circuits (the defining channels for the components is assumed to be given). Let $\mathcal{H}_{ij}$ be

an entry of a stochastic matrix. Then $\mathcal{E}_{ij} = 1 - \mathcal{H}_{ij}$ is the corresponding entry of the error matrix and $P_{ij} = -\ln(1 - \mathcal{H}_{ij})$ is the corresponding entry of the reliability matrix. If the entries of the reliability matrix are not too dispersed then $P_0 = \min P_{ij}$ is a useful measure of reliability for the sequential circuit.

4. *Future Trends*:

4.1. *Preliminaries*: Section 4 of the paper treats what the author considers as future trends associated with the theory of switching. The future trends in the theory of switching are treated in two parts: 4.2. trends in theory and 4.3. trends in applications.

4.2. *Trends in Theory*:

4.2.1. *Preliminaries*: The theory of section 4.2 is treated in four parts: 4.2.2. algebra, 4.2.3. mathematical logic, and 4.2.4. mathematical analysis and 4.2.5. topology.

4.2.2. *Algebra*: Further work will continue on the application of the more recent results of lattice theory to the analysis and synthesis of asynchronous switching circuits. In this respect, the works of Muller [34] will be a most useful foundation.

The use of Galois theory to study the general properties of delay polynomials [47] should produce some interesting results.

4.2.3. *Mathematical logic*: New schemes of characterizing switching circuits, such as Lee's [41] binary decision programs, will be sought. Recursive function theory should be useful in this respect (and incidently making a measure of complexity for Boolean function meaningful). More work will be done with the decision problem associated with automata.

4.2.4. *Mathematical Analysis*: More work will be done to determine the properties of the class of error-reducing configurations. Mathematical tools will be developed (or rediscovered) so as to handle the case of potential-level stochastic switching circuits.

4.2.5. *Topology*: Further applications of algebraic topology to the synthesis of switching circuits will be made. In addition, more use will be made of combinatorial topology in the synthesis of switching circuits.

4.3. *Trends in Applications*:

4.3.1. *Preliminaries*: The trends in applications are treated in three parts: 4.3.2. information transduction, 4.3.3. adaptive control systems and 4.3.4. synthetic intelligence.

4.3.2. *Information Transduction*: More use will be made of the basic properties of switching networks to provide error detection and correction of transmitted messages.

4.3.3. *Adaptive control systems*: Switching systems will play an important role in the analysis and synthesis of control systems capable of adapting to themselves [60,61].

4.3.4. *Synthetic intelligence*: Switching systems will play an important role when one is analyzing and synthesizing systems that show of the intuitive properties associated with intelligence [62,63].

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3

On Critical Shaft Speeds in Flexure of Spring Supported Automotive Drive Lines

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The design of vehicular propeller shafts requires careful consideration of its dynamic behavior in flexure in addition to meeting manufacturer's specifications. The need for a comprehensive study of critical shaft speeds led to a research contract awarded to Wayne State University's College of Engineering by the Dana Corporation of Toledo, Ohio. The primary objective of this investigation is to determine the influence of elastic supports on the shaft's free vibration frequency.

The frequency study of rotating shafts is predicated on the following assumptions:

1. Shafts have a uniform and circular cross-section.
2. The material is isotropic.
3. Only flexural oscillations are considered, independent of torsional and longitudinal vibrations.
4. The familiar relationship between shaft curvature and bending moment ($y'' = -\frac{M}{EI}$) is valid.
5. The shaft is ideally balanced.
6. Rotation of small shaft elements because of flexure and shear effects are neglected.
7. The shaft is free of torque.

The following notations are used throughout the article:

x, y, z	Cartesian co-ordinates
ω	Circular frequency of shaft rotation
$\bar{\omega}_n$	Circular frequency of bounce motions
ω_n	Circular natural frequency of flexure
$\bar{\bar{\omega}}_n$	Circular frequency of pitch motion
I	Inertia moment of shaft cross-section
E	Young's Modulus
γ	Specific Weight

g	Gravity constant
μ	Shaft mass per unit length
t	Time
L	Shaft length
a	Argument of elastica per unit length
$(aL) = P$	Argument of elastica
k_1, k_2	Linear spring rate of supports
$K = k_1 + k_2$	Resultant linear spring rate
kb_1, kb_2	Rate of flexural restraint at supports
$Kb = kb_1 + kb_2$	Resultant rate of flexural restraint at supports
$\rho^2 = \frac{EI}{kL^3}$	Stiffness ratio of beam $(\frac{EI}{L^3})$ and elastic supports (k)
M_x	Bending moment at x
V_x	Shear force at x
Q_x	Rate of change of shear force. (Also dynamic loading rate)

Introduction

The fundamental frequencies of vibrating beams having uniform and symmetrical cross-section may be obtained by equating the elastic restoring forces within the beam to the inertia force of mass at any arbitrary distance X from the support. The equations of motion of a shaft portion of unit length are:

$$\begin{aligned} I_{zx}\, E \frac{\delta^4 y}{\delta x^4} + \mu \frac{\delta^2 y}{\delta t^2} &= 0 \\ I_{yx}\, E \frac{\delta^4 z}{\delta x^4} + \mu \frac{\delta^2 z}{\delta t^2} &= 0 \end{aligned} \qquad (1)$$

In Equations (1) the first terms denote the rate of change of the shear force Q_x and the second term, the inertia load per unit length of shaft. A product solution of the form

$$y = F_1(x)\phi(t) \qquad z = F_2(x)\phi(t)$$

will satisfy the above partial differential equation. Since the shafts under consideration have a circular cross-section it will suffice to consider but one of Equations (1).

A particular solution is known to be:

$$Y = \sum^{n} (C_1 \sin h\, a_n x + C_2 \cos h\, a_n x + C_3 \sin a_n x + C_4 \cos a_n x)(\sin \omega_n t + \cos \omega_n t) \qquad (2)$$

where C_1, through C_4 are arbitrary constants to be determined from the conditions at the shaft supports. If we choose simple supports the following boundary conditions must be satisfied:

$$y_{x=0} = 0 \quad y_{x=1} = 0 \quad \frac{\delta^2 y}{\delta x^2}\Big|_{x=0} = 0 \quad \frac{\delta^2 y}{\delta x^2}\Big|_{x=1} = 0 \tag{3}$$

It can be shown that all arbitrary constants but C_3 must become zero in order to fulfill above requirements (Eq. 3) and hence the solution reduces to:

$$y_n = C_3 \sin a_n x \, (\sin \omega_n t + \cos \omega_n t) \tag{2a}$$

and with a suitable choice of initial conditions this may be further reduced to:

$$y_n = C_3 \sin a_n x \sin \omega_n t \tag{2b}$$

for:

$$y_{x=1} = 0 \quad aL = \pi, \; 2\pi \cdots n\pi \tag{4}$$

where n is an integer denoting the n th noded mode of vibration. Substitution of Eq. (2b) in Eq. (1) yields the frequency equation:

$$\omega_n^2 = \frac{IE}{\mu} a^4$$

and since

$$a = \frac{n\pi}{L}$$

$$\omega_n^2 = \frac{IE\pi^4 n^4}{\mu L^4}$$

for n = 1 we obtain the fundamental frequency:

$$\omega_n = \frac{\pi^2}{L^2}\sqrt{\frac{IE}{\mu}} \tag{5}$$

Expression (5) will assume the form of:

$$\omega_n = \frac{p^2}{L^2}\sqrt{\frac{IE}{\mu}} \quad \text{where} \quad p \neq \pi \tag{5a}$$

for boundary conditions which are different from the assumed ones. Hence, the parameter p expressing the argument of the elastic line is dependent on the conditions of constraint at the shaft supports. The above expressions (5) are reasonably well known and yield nearly exact values of the fundamental shaft frequency (1,4).

Beams and shafts on elastic supports (2,3)

In general, drive shafts rest on supports assumed to be rigid. If the shaft rests on very weak elastic supports, the natural frequency of shaft bounce motion can be determined from the relation:

$$\bar{\omega}_n = \sqrt{\frac{K}{m}} \tag{6}$$

whereby: $m = \mu L$ = Shaft Mass

For pitch motion we obtain:

$$\bar{\omega}_n = \sqrt{3}\sqrt{\frac{K}{m}} \tag{6a}$$

Shaft assemblies which include universal or Hooke joints were found to flex between the joint yokes (the joint permits unrestrained bending of shaft) if the shaft rigidity is considerably lower than that of the joints. However, vibrations of propeller shafts having a spring rate of approximately the same magnitude than that of the Hooke joints will cause flexural vibrations in the joints also. The exact determination of the effective shaft length "L" which exceeds that of the shaft proper is not always possible and will depend on the degree of rigidity of the supports. Hence, the elasticity of the supporting structure must be expressed in the boundary equations.

We shall consider two distinct boundary problems, namely:

1. Shaft supported on springs of rate k_1 and k_2 (See Figure 1).
2. Shaft systems according to (1) having also elastic constraints of rate k_{b_1} and k_{b_2} at their ends (See Figure 2).

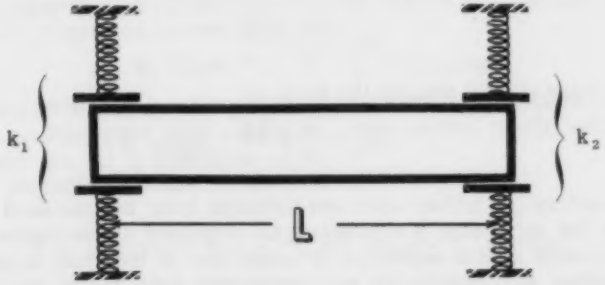

Figure 1.

It is clearly seen that the boundary conditions for Figure 1 require that:

$$M_{x=0} = 0 \quad \text{or} \quad IE \left.\frac{\delta^2 y}{\delta x^2}\right|_{x=0} = 0$$

$$M_{x=1} = 0 \quad IE \left.\frac{\delta^2 y}{\delta x^2}\right|_{x=1} = 0$$

also: (7)

$$y_{x=0} \times k_1 = -IE \left.\frac{\delta^3 y}{\delta x^3}\right|_{x=0} = V_{x=0}$$

$$y_{x=1} \times k_2 = IE \left.\frac{\delta^3 y}{\delta x^3}\right|_{x=1} = V_{x=1}$$

Upon substitution of the particular solution (Eq. 2) and its second and third derivatives with respect to "X" in Eq. (7) we obtain four homogeneous algebraic equations, in C_1, C_2, C_3, C_4, viz.:

$$a^3 + \frac{k_1 C_2 + C_4}{EI C_1 + C_3} = 0 \qquad \text{a)}$$

$$a^2 (C_2 - C_4) = 0 \qquad \text{b)}$$

$$C_1 \sin h\, aL + C_2 \cos h\, aL - C_3 \sin aL + C_4 \cos aL = 0 \qquad \text{c)} \qquad (8)$$

$$C_1 (a^3 EI \cos h\, aL - k_2 \sin h\, aL) + C_2 (a^3 EI \sin h\, aL - k_2 \cos h\, aL)$$

$$- C_3 (a^3 EI \cos aL + k_2 \sin aL) + C_4 (a^3 EI \sin aL - k_2 \cos aL) = 0 \qquad \text{d)}$$

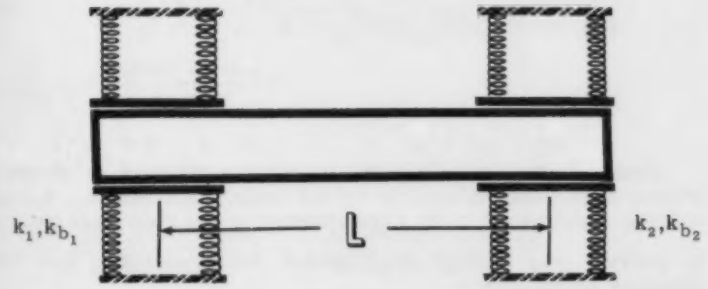

Figure 2.

Since the characteristic values of the vibrating beam are independent of the initial conditions we may choose one arbitrary constant, say $C_3 = 1$. Writing the equations in dimensionless form, adopting the following notation:

$$aL = p \qquad \rho_1^2 = \frac{EI}{k_1 L^3} \qquad \rho_2^2 = \frac{EI}{k_2 L^3}$$

the remaining arbitrary constants C_1, C_3, C_4 are found from Equations (8b, 8c, 8d):

$$C_4 = 1$$

$$C_3 = \frac{p^3\rho_2^2 [\coth(\cosh p - \cos p) - \sinh p - \sin p] + \cos p}{p^3\rho_2^2 [\sin p \coth p - \cos p] - \sin p}$$

$$C_1 = \frac{p^3\rho_2^2 [\coth p(\cosh p - \cos p) - \sinh p - \sin p] + \cos p}{p^3\rho_2^2 [\cosh p - \cot p \sinh p] - \sinh p} \tag{9}$$

$$+ \frac{\cos p - \cosh p}{\sinh p}$$

Upon substitution of the expression of the arbitrary constants in Equation (8a) one obtains the frequency equation in terms of the parameter p (argument of the elastica) as a function of stiffness ratio ρ^2, that is:

$$\frac{p^3\rho_2^2 [\coth p(\cosh p - \cos p) - \sinh p - \sin p] + \cos p}{p^3\rho_2^2 [\cosh p - \cot p \sinh p] - \sinh p}$$

$$- \frac{p^3\rho_2^2 [\coth p(\cosh p - \cos p) - \sinh p - \sin p] + \cos p}{p^3\rho_2^2 [\sin p \coth p - \cos p] - \sin p} \tag{10}$$

$$+ \frac{\cos p - \cosh p}{\sin h p} = -\frac{1}{p^3\rho_1^2}$$

Next we shall consider the boundary conditions as shown in Figure 2 whereby, in addition to the support springs (k_1, k_2) two bending restraints (k_{b_1}, k_{b_2}) are attached at the shaft ends such as to prevent free angular displacement during flexure. The four boundary equations are:

$$y_{x=0} \times k_1 = -\left.\frac{\delta^3 y}{\delta x^3}\right|_{x=0} EI \qquad \left.\frac{\delta y^2}{\delta x_2}\right|_{x=0} EI = \left.\frac{\delta y}{\delta x}\right|_{x=0} k_{b_1}$$

$$y_{x=1} \times k_2 = \left.\frac{\delta^3 y}{\delta x^3}\right| EI \qquad \left.\frac{\delta^2 y}{\delta x^2}\right|_{x=1} EI = \left.\frac{\delta y}{\delta x}\right|_{x=1} k_{b_2} \tag{11}$$

After differentiating the solution (Eq. 2) once, twice and three times and upon substitution in Eq. 11 we obtain again four homogeneous algebraic equations, viz.:

$$a^3 + \frac{k_1}{EI}\frac{C_2 + C_4}{C_1 - C_3} = 0 \qquad \text{a)}$$

$$C_1 (a^3 EI \cosh aL - k_2 \sinh aL) + C_2 (a^3 EI \sinh aL - k_2 \cosh aL)$$
$$- C_3 (a^3 EI \cos aL + k_2 \sin aL) + C_4 (a^3 EI \sin aL - k_2 \cos aL) = 0 \qquad \text{b)}$$

$$a - \frac{k_{b_1}}{EI}\frac{C_1 + C_3}{C_2 - C_4} = 0 \qquad \text{c)} \tag{12}$$

$$C_1 (k_{b_2} \cosh aL + EI\, a \sinh aL) + C_2 (k_{b_2} \sinh aL + EI\, a \cosh aL)$$
$$+ C_3 (k_{b_2} \cos aL - EI\, a \sin aL) - C_4 (k_{b_2} \sin aL + EI\, a \cos aL) = 0 \qquad \text{d)}$$

Equations (12) reduce to a single equation in p,

$$\rho_1^2 = \frac{EI}{k_1 L^3} \quad \rho_2^2 = \frac{EI}{k_2 L^3} \quad \rho_{b_1}^2 = \frac{EI}{k_{b_1} L} \quad \rho_{b_2}^2 = \frac{EI}{k_{b_2} L}$$

For the purpose of brevity the final modal equation is presented without the intermediate steps. Then,

$$p^4 \rho_1^2 \rho_{b_1}^2 = -\frac{DQ_1 + GQ_2 - 2p\rho_{b_1}^2 Q_3}{GQ_4 + DQ_5} \tag{13}$$

where

$$Q_1 = F - E \quad Q_2 = A + B \quad Q_3 = AF + EB$$
$$Q_4 = B - A \quad Q_5 = F + E$$

$$A = \frac{p^3 \cosh p - 1/\rho_2^2 \sinh p}{p^3 \tanh p - 1/\rho_2^2} - \frac{1/\rho_{b_2}^2 \cosh p + p \sinh p}{1/\rho_{b_2}^2 \tanh p - p}$$

$$B = \frac{p^3 \cos p + 1/\rho_2^2 \sin p}{p^3 \tanh p - 1/\rho_2^2} + \frac{1/\rho_{b_2}^2 \cos p - p \sin p}{1/\rho_{b_2}^2 \tanh p + p}$$

$$D = \frac{p^3 \sin p - (\frac{1}{\rho_2^2}) \cos p}{p^3 \tanh p - 1/\rho_2^2} + \frac{1/\rho_{b_2}^2 \sin p + p \cos p}{1/\rho_{b_2}^2 \tanh p + p}$$

$$E = \sin h\, p\, (\frac{p^2}{\rho_{b_1}^2} - \frac{p}{\rho_2^2}) + \cos h p\, (p^4 - \frac{1}{\rho_{b_1}^2 \rho_2^2})$$

$$F = \sinh p\, (\frac{p^3}{\rho_{b_1}^2}) - \frac{p}{\rho_2^2} \sin p - p^4 \cos p - \frac{1}{\rho_{b_1}^2 \rho_2^2} \cos h p$$

$$G = p^4 (\sinh p + \sin p) - \frac{p}{\rho_2^2} (\cos h p + \cos p)$$

As one can readily recognize Equation (13) is a transcendental algebraic equation containing the parameter "p" in an implicit form. "p" is found by an iterative method executed on a digital computer. Its value depends solely on the magnitude of the chosen stiffness ratios ρ_1^2 ρ_2^2 $\rho_{b_1}^2$ $\rho_{b_2}^2$ which, again, are functions of E, I, L and k_1, k_2, k_{b_1} k_{b_2}.

DISCUSSION OF THE FREQUENCY EQUATIONS

Both equation (10) and equation (13) yield infinitely many solutions for "p". For a trial solution, the value of $\rho^2 = .1$ is assumed. The following results are obtained for the first modes of a system as illustrated in Fig. 1. We obtain the following wave lengths:

First mode	p = 1.74
Second mode	p = 2.33
Third mode	p = 4.82
Fourth mode	p = 7.88

It is not a coincidence that the difference between two successive values of p appear to approach π for modes above the third one. The hyperbolic terms "cos h" and "sin h" both tend towards infinity and "cot h" towards 1 for increasing values of P. Hence, their influence on the frequency equation for the higher modes diminishes. The results of equations (10) and (13) are plotted in Fig. 3 and Fig. 4

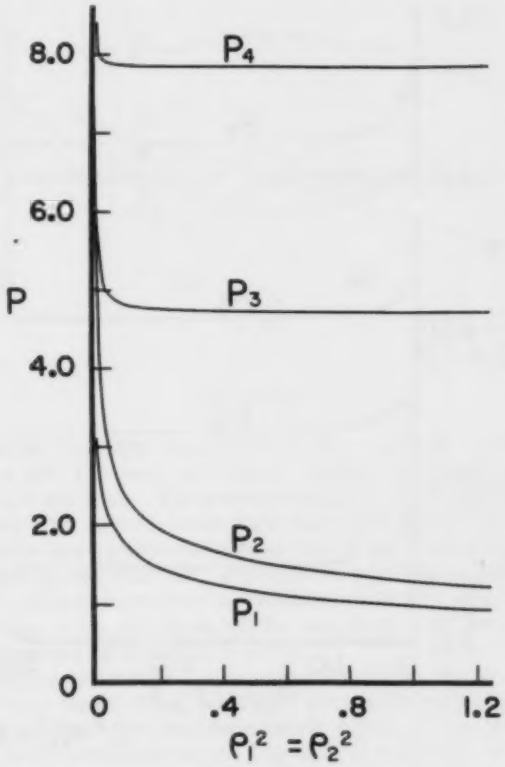

Figure 3.

respectively. Wave-lengths p corresponding to modes 1, 2, 3, and 4 are plotted versus stiffness ratio ρ^2. In Fig. 4 we show again wave-length p versus ρ_b^2, the stiffness ratio with respect to the bending restraint, hence, we refer to Fig. 2. In order to plot the curves we assume $\rho^2 = .01$ = constant. With this assumption the shear restraint is given and only the bending restraint is variable. Of course once the wave-length p is determined we can substitute the value immediately into equation (5a) and obtain the circular frequency of flexure (See Fig. 9). Dividing the value by 2π we obtain the frequency in cycles per second. In the following we shall interpret the meaning of Fig. 3.

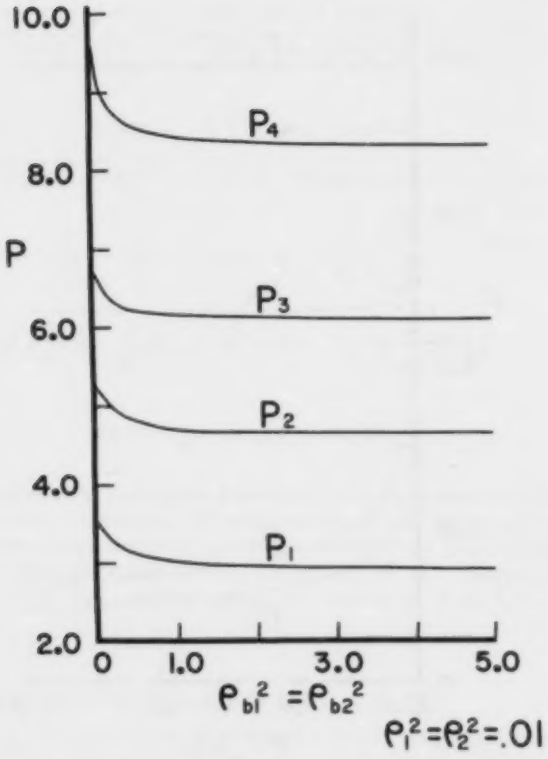

Figure 4

It is seen that for extremely small values of p, say less than .05 the curve is rising steeply. Small values of the rigidity ratio imply very high stiffness values of the support springs. If we approach the limit which is zero we should obtain the frequencies which correspond to those of a simply supported beam. On the other hand if the rigidity ratio ρ^2 assumes an infinite value the condition implies supports of zero spring rates or, we may say, the supports are eliminated. It is seen that the first two modes expressed by wave-lengths p_1 and p_2 approach zero, that is they vanish. However, mode three and following approach a minimum value, e.g. mode three approximately 4.68 which corresponds to the wave-length of a "free-free" beam, or an unsupported beam. Let us consider the first two modes for weak elastic supports, that is, spring mounts

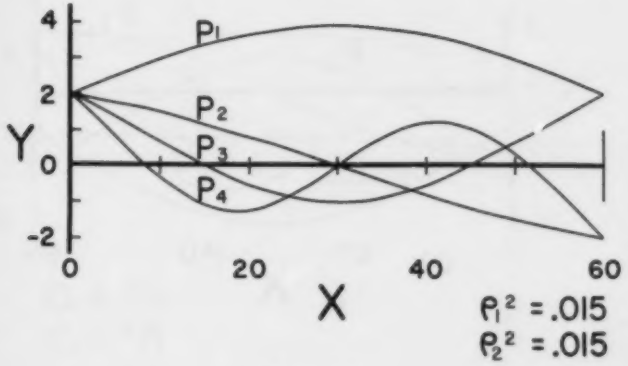

Figure 5.

which result in a rigidity ratio of $\rho^2 \geq .10$. The first flexural mode degenerates into the state of bounce motion, the shaft being considered as a rigid body. The second flexural mode degenerates into pitch motion. Yet, in both cases light flexure is thought to be superimposed. over the motion pattern of the system as a rigid body. Hence, we recognize that the existence of external constraints increases the possible number of vibration modes by two, if we exclude motions in the direction of the length axis of the beam, and restrict ourselves to motion in a plane.

It is advantageous to plot the deflection curves of the shafts for each particular mode. Thus, we obtain the configuration of the beam during vibration and we can immediately associate the mode to the number of node points obtained. The elastic lines for those modes which pertain to simple constraints (Fig. 1) are illustrated in Fig. 5. Those pertaining to the combined shear and bending constraints are illustrated in Fig. 6. From these two illustrations we can immediately conclude the effect of relaxation of constraints and observe that the two lowest modes of flexure degenerate, in effect to vibration modes corresponding to bounce and pitch motion of the shaft as a rigid body.

It remains to be seen what effect unsymmetrical constraints would have on the frequency of the system. Let us assume that the left hand constraints $\rho^2_{b_1}$ and ρ^2_1, are unequal to the right hand constraints $\rho^2_{b_2}$ and ρ^2_2. In general the frequencies observed tend more toward those values which correspond to the lower rigidity ratios regardless of the position of the weak elastic support with respect to the midpoint of the shaft. The elastic line of vibrating shafts

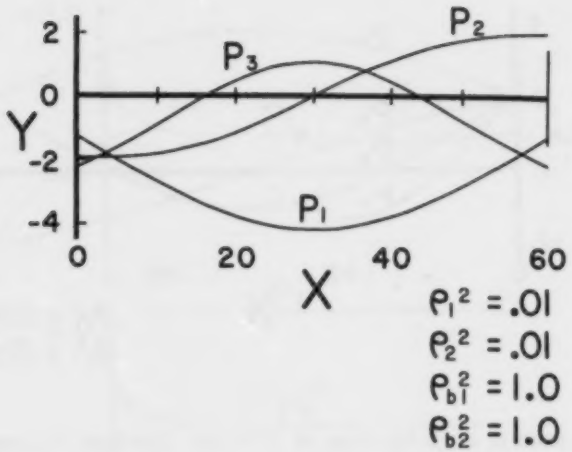

Figure 6.

having unsymmetrical constraints is shown in Fig. 7, 7a, and Fig. 8. Figures 7 and 7a pertain to a shaft with unsymmetrical constraints, for example $\rho_1^2 = .025$, $\rho_2^2 = .015$ and $\rho_1^2 = .015$, $\rho_2^2 = .025$. Figure 8 corresponds to unsymmetrical bending constraints, namely $\rho_{b_2} = 500$, $\rho_{b_1}^2 = 50$ with constant shear constraints $\rho_1^2 = \rho_2^2 = .01$. As

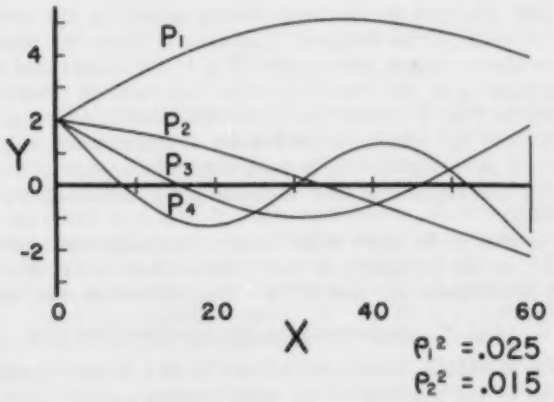

Figure 7.

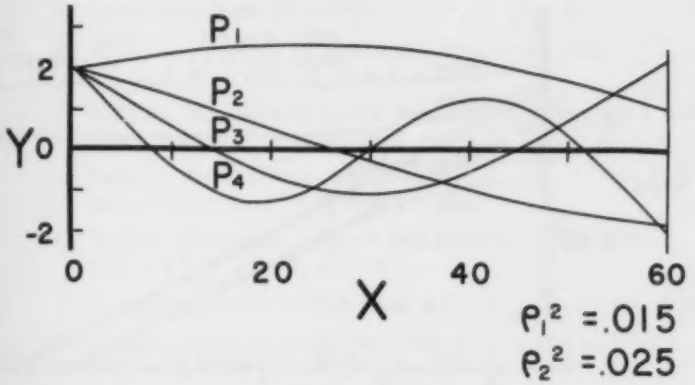

Figure 7a.

was expected the asymmetry of the support springs is reflected in the elastic line of the vibrating shafts. In all cases discussed, the condition of orthogonality of the natural modes, namely

$$\int_0^1 y_n(x)\, y_m(x)\, dx = 0$$

must hold.

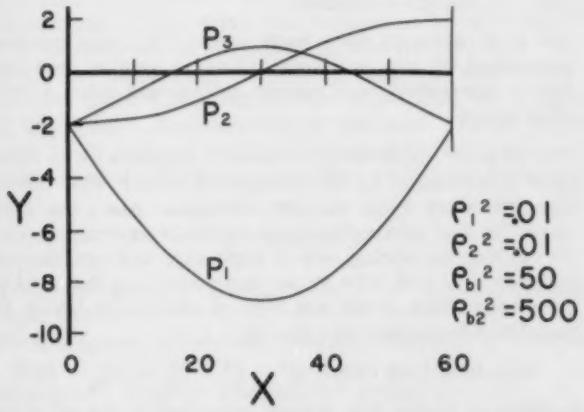

Figure 8.

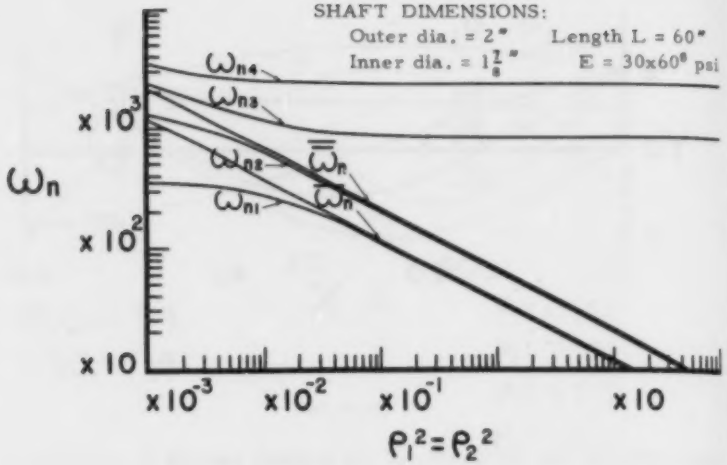

Figure 9.

SUMMARY OF RESULTS

In conclusion of our discussion four important facts have been established:

1. Elastic supports lower the natural frequency of beams subject to flexural oscillation.
2. The application of very weak support springs causes a degeneration of the two lowest flexure modes into vibration modes associated with bounce motion and pitch motion of a rigid mass.
3. The effect of asymmetry of elastic supports is to reduce the natural frequency as to correspond rather with the weaker, than the more rigid support stiffness. Also, the deflection curve for any particular mode becomes asymmetrical.
4. By varying the spring rate of right hand and left hand support between zero and infinity we can determine the natural frequencies of beams for any type of constraint which is contemplated including the following:
   a. Built-in at both ends. ($\rho_1^2 = \rho_2^2 = \rho_{b_1}^2 = \rho_{b_2}^2 = 0$)
   b. Built-in at one end, simply supported at the other end ($\rho_{b_1}^2 = \rho_2^2 = \rho_1^2 = 0$, $\rho_{b_2}^2 = \infty$)

c. Cantilever beam ($\rho^2_{b_1} = \rho^2_1 = \infty$, $\rho^2_{b_2} = \rho^2_2 = 0$)

d. Built-in, but movable at one end, free at the other ($\rho^2_{b_1} = 0$ $\rho^2_1 = \rho^2_2 = \rho^2_{b_2} = \infty$)

e. Built-in, but movable at one side, simply supported at the other ($\rho^2_{b_1} = \rho^2_2 = 0$, $\rho^2_{b_2} = \rho^2_1 = \infty$)

f. Built-in, but movable at both sides ($\rho^2_{b_1} = \rho^2_{b_2} = 0$, $\rho^2_1 = \rho^2_2 = \infty$)

g. Built-in at one end, built-in and movable at other side ($\rho^2_{b_1} = \rho^2_{b_2} = \rho^2_1 = 0$, $\rho^2_2 = \infty$)

h. Simply supported at both ends ($\rho^2_{b_1} = \rho^2_{b_2} = \infty$ $\rho^2_1 = \rho^2_2 = 0$)

i. Simply supported at one end, free at the other end ($\rho^2_{b_1} = \rho^2_{b_2} = \rho^2_1 = \infty$, $\rho^2_2 = 0$)

j. Free-free beam ($\rho^2_{b_1} = \rho^2_{b_2} = \rho^2_1 = \rho^2_2 = \infty$)

Theoretical consideration and experimental evidence point to the desirability to suspend drive lines on elastic mountings. This will lower the critical vibration frequencies of flexure and will allow operation through these regions of instability in a virtually vibration-free manner. A forthcoming publication during 1961 of Soc. of Auto. Eng. will discuss the experimental aspects of this work in detail.

ACKNOWLEDGMENTS

The author is indebted to the Dana Corporation, Toledo, Ohio for their sponsorship of the drive line research. In particular he owes thanks to their Director of Research and Development, Mr. Philip Mazziotti, who made many helpful suggestions, and to Mr. Robert Warnke, also Dana Corporation, who undertook the programming work of the problem and execution of the same. Finally, thanks are expressed to all persons who were consulted during the research and contributed directly or indirectly in the compilation of the material.

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4

Applying a Matrix Partitioning Technique to Solving Steady State Vibration Problems*

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ABSTRACT

This paper describes a matrix partitioning procedure which can be used for solving large systems of linear equations which have a large quantity of zero terms. The use of a control matrix is introduced to save solution time for an example vibration problem. Further gains are experienced by rearrangement of the matrices as demonstrated on a sample problem. The techniques discussed have made possible solutions to problems which otherwise could not be handled on available computers.

In vibration work, many systems can best be described by point masses connected together by weightless springs. A single mass in a system can move in six possible directions and therefore, requires six simultaneous equations to describe it. To simulate a complex system would require many mass points. For instance, a car body cannot be properly described with less than about 70 interconnected point masses. Using the assumptions that the system is nearly linear and that steady state results are required, the car body would require 420 simultaneous linear equations to describe it.

Digital computers lend themselves to solutions of linear equations through the use of matrix operations. However, with limited storage capacity and the speed of operation of present computers, direct inversion of matrices of the order previously discussed are uneconomical. It is therefore necessary to use techniques which accomplish the solution through a method other than direct inversion. One solution which has been previously used is a partitioning technique[1]. Through the use of this method, advantage can be taken of the zero terms to cut down on the solution time.

Partitioning is a process of dividing a large matrix into several smaller matrices. The process of solving a set of linear equations

*I wish to acknowledge the assistance given by Mr. Fred Timpner of Pontiac Motor Division on both the original problem and on this paper.

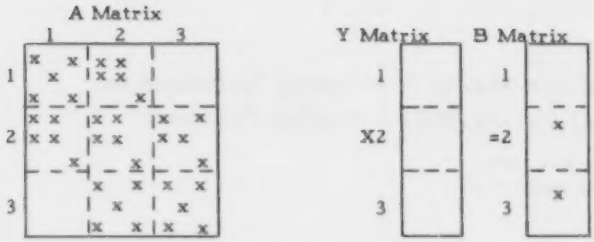

Figure 1. An Example of Matrix Partitioning

using the partitioning technique can be explained using the example equations shown in Figure 1. In the sample problem, the x's represent non-zero terms. The partitioning shown divided the large A matrix into square submatrices. However, this is not necessary for applying the partitioning technique. The equations in Figure 1 can be represented by the three following equations where each term represents a submatrix.

1a.) $A(1, 1) \times Y(1) + A(1, 2) \times Y(2) + A(1, 3) \times Y(3) = B(1)$

1b.) $A(2, 1) \times Y(1) + A(2, 2) \times Y(2) + A(2, 3) \times Y(3) = B(2)$

1c.) $A(3, 1) \times Y(1) + A(3, 2) \times Y(2) + A(3, 3) \times Y(3) = B(3)$

The solution for Y can be obtained in the following manner.

The first restriction on the solution is that the matrix A(1, 1) must be of such a nature that its inverse can be found. On the assumption that an inverse, $A(1, 1)^{-1}$, can be found for A(1, 1), the first step would be to multiply equation 1a.) by the inverse matrix $A(1, 1)^{-1}$, to get equation 2a.).

2a.) $A(1, 1)^{-1} \times A(1, 1) \times Y(1) + A(1, 1)^{-1} \times A(1, 2) \times Y(2)$
$+ A(1, 1)^{-1} \times A(1, 3) \times Y(3) = A(1, 1)^{-1} \times B(1)$

When equation 2a.) is multiplied by matrix A(2, 1), the result will be equation 2b.). Likewise, multiplying equations 2a.) by the matrix A(3, 1) will yield equation 2c.).

2b.) $A(2, 1) \times Y(1) + A(2, 1) \times A(1, 1)^{-1} \times A(1, 2) \times Y(2) + A(2, 1)$
$\times A(1, 1)^{-1} \times A(1, 3) \times Y(3) = A(2, 1) \times A(1, 1)^{-1} \times B(1)$

2c.) $A(3, 1) \times Y(1) + A(3, 1) \times A(1, 1)^{-1} \times A(1, 2) \times Y(2) + A(3, 1)$
$\times A(1, 1)^{-1} \times A(1, 3) \times Y(3) = A(3, 1) \times A(1, 1)^{-1} \times B(1)$

The three equations 2a.), 2b.) and 2c.) can be developed only if the matrices are of correct size for multiplication. This is the second restriction on the solution. Providing these restrictions have been met, equation 2b.) can be subtracted from equation 1b.) to give equation 3a.). In a similar manner, equation 2c.) can be subtracted from equation 1c.) to yield equation 3b.).

$$3a.)\ [A(2,2) - A(2,1) \times A(1,1)^{-1} \times A(1,2)] \times Y(2) + [A(2,3) - A(2,1) \times A(1,1)^{-1} \times A(1,3)] \times Y(3) = B(2) - A(2,1) \times A(1,1)^{-1} \times B(1)$$

$$3b.)\ [A(3,2) - A(3,1) \times A(1,1)^{-1} \times A(1,2)] \times Y(2) + [A(3,3) - A(3,1) \times A(1,1)^{-1} \times A(1,3)] \times Y(3) = B(3) - A(3,1) \times A(1,1)^{-1} \times B(1)$$

The above equations can be simplified by the following substitutions:

$$C(2,2) = A(2,2) - A(2,1) \times A(1,1)^{-1} \times A(1,2)$$
$$C(2,3) = A(2,3) - A(2,1) \times A(1,1)^{-1} \times A(1,3)$$
$$C(3,2) = A(3,2) - A(3,1) \times A(1,1)^{-1} \times A(1,2)$$
$$C(3,3) = A(3,3) - A(3,1) \times A(1,1)^{-1} \times A(1,3)$$
$$D(2) = B(2) - A(2,1) \times A(1,1)^{-1} \times B(1)$$
$$D(3) = B(3) - A(3,1) \times A(1,1)^{-1} \times B(1)$$

After making the above substitutions, the equations appear as follows:

$$3a.)\ C(2,2) \times Y(2) + C(2,3) \times Y(3) = D(2)$$

$$3b.)\ C(3,2) \times Y(2) + C(3,3) \times Y(3) = D(3)$$

Providing that the matrix C(2, 2) has an inverse, the equation 3a.) can be multiplied by the inverse matrix, $C(2,2)^{-1}$, to obtain equation 4a.).

$$4a.)\ C(2,2)^{-1} \times C(2,2) \times Y(2) + C(2,2)^{-1} \times C(2,3) \times Y(3) = C(2,2)^{-1} \times D(2)$$

When equation 4a.) is multiplied by the matrix C(3, 2), the result is equation 4b.).

4b.) $C(3, 2) \times Y(2) + C(3, 2) \times C(2, 2)^{-1} \times C(2, 3) \times Y(3)$
$= C(3, 2) \times C(2, 2)^{-1} \times D(2)$

Subtracting equation 4b.) from equation 3b.) will give the following equation

5a.) $[C(3, 3) - C(3, 2) \times C(2, 2)^{-1} \times C(2, 3)] \times Y(3)$
$= D(3) - C(3, 2) \times C(2, 2)^{-1} \times D(2)$

When the following substitutions are made, the result will be equation 5b.).

$$E(3, 3) = C(3, 3) - C(3, 2) \times C(2, 2)^{-1} \times C(2, 3)$$
$$F(3) = D(3) - C(3, 2) \times C(2, 2)^{-1} \times D(2)$$

5b.) $E(3, 3) \times Y(3) = F(3)$

From equation 5b.), the matrix Y(3) can be obtained. One method by which this can be accomplished is as follows. First, it is necessary to obtain the inverse of the matrix E(3, 3). Then, by multiplying the matrix F(3) by the inverse of matrix E(3, 3), the matrix Y(3) can be obtained. The matrix Y(2) can now be obtained from equation 6a.) by substituting in the matrix Y(3) and solving. Finally, the matrix Y(1) can be obtained by substituting matrices Y(2) and Y(3) in equation 1a.).

One important point needs emphasizing at this time. Notice that the C matrices can be placed over the A matrices without destroying an A matrix that will be needed later on during this solution. Likewise, the D matrices can be placed over the B matrices, the E matrices over the C matrices, and the F matrices over the D matrices. This means that no additional space will be needed for the solution of the equations.

There seems to be no advantage to this solution over an inversion solution until the submatrices containing all zero terms are considered. Although there are only 2 submatrices out of 9 which contain all zero terms, the number of matrix multiplications in the example problem drops from 17 to 11. However, it would be necessary to somehow know before the solution was started which submatrices were composed of all zero terms. One method of accomplishing this would be to check all the submatrices for zero terms and set up a table. As an example, a table could be constructed for the sample problem in Figure 1. The ones in the table will represent submatrices which do not have all zero terms and the

zeros will represent submatrices which have all zero terms. The table or control matrix for the A submatrices would appear as follows:

	1	2	3
1	1	1	0
2	1	1	1
3	0	1	1

Figure 2. Control Matrix for A Submatrices

By first looking in the control matrix, it is now possible to tell if a matrix multiplication is necessary. With the aid of the control matrix, it is now possible to cut down the time required for solution.

In the application of partitioning to vibration problems, there can be considerable more savings. For instance, the control matrix previously described can perform an additional function. The zero terms in the control matrix indicate that no multiplications by the submatrix represented would be required. Therefore, there is no need to store the submatrix represented by the zero term. However, the control matrix must now take on the function of locating the storage area of the non-zero submatrices. This can be accomplished by storing the submatrices in a three dimensional array where the third subscript is used for location purposes. By storing the third subscript in the control matrix, the presence of a number greater than zero indicates that multiplication is necessary and tells where in storage to locate the submatrix. For example, the control matrix for the problem in Figure 1 might appear as follows:

	1	2	3
1	1	2	0
2	3	4	5
3	0	6	7

Figure 3. Control Matrix Showing Storage Assignment

Using this arrangement, the storage required can be reduced for a problem, such as vibrations, containing some zero submatrices.

At this point, a review of how the complete matrix is built up from the data for a vibration problem seems in order. The forces that a spring exerts cause two mass points to be coupled together and exert resistance to the movement of the mass points. If each mass is given the maximum of six degrees of freedom, the coupling terms will be contained in two six by six matrices which will be off the main diagonal. The resistance to mass movement terms will be contained in two six by six matrices, one for each mass point, which will lay along the diagonal of the matrix for the entire system. Since all the terms are contained in six by six matrices, the optimum size of submatrix for the system appears to be six by six. Using this size of submatrix allows storage of the fewest zero terms in the submatrices without getting the required control matrix too large and, also allows direct setup of the control matrix. There are two advantages of setting up the control matrix as the equations are developed. First, no searching is required later on to determine which submatrices are all zero and secondly, the complete matrix is never developed so the storage requirements for a given problem are lowered.

When no storage is allowed for submatrices which contain all zero terms, there is a new problem introduced into the solution. For example, it is possible to end up with a control matrix similar to the one in Figure 4, where the numbers represent the non-zero

	1	2	3	4	5	6
1	1		2		15	
2		5	X		6	
3	3	X	4		X	12
4				9	10	X
5	16	7	X	11	8	X
6			13	X	X	14

Figure 4. Sample Control Matrix

submatrices. During the process of solution, it will be found necessary to allocate storage for newly formed submatrices. The new submatrices required for the sample control matrix shown in Figure 4 are indicated by the X's. A general rule for determining in advance the additional storage required can be stated as follows. For non-zero submatrices above the diagonal, the submatrices in the column between the non-zero terms and the diagonal will be required to complete the solution. For non-zero submatrices below the diagonal, the submatrices in the row between the non-zero terms and the diagonal terms will be required for the solution. Since these additional submatrices required for the solution cut down on the storage capacity and increase the solution time for a given problem, there is a need to keep them down to a minimum.

One method of accomplishing a reduction in the number of additional submatrices required for the solution is through the use of row and column interchanges. Advantageous interchanges can be recognized easily when a matrix is being rearranged by hand. However, letting a digital computer do the rearranging requires some organized method. In the following paragraphs, a description of the process used at Pontiac for selecting row and column interchanges will be described.

The first step in arriving at the necessary interchanges is to make up a table showing the coupling terms connected to each point. Using Figure 4 as an example problem, the table would appear as shown in Figure 5.

Point	Coupling Terms
1	3,5
2	5
3	1,6
4	5
5	1,2,4
6	3

Figure 5. Coupling Table

Using this table, a new order for the points can be developed. The new order is developed by taking the point number from the table and following the point number by its coupling terms. Before a number is added to the new table, a check is made to see if it already appears in the table. If it does, the number is not added to the new table. The new table developed from Figure 5 would appear as the first down column on left side of Figure 6.

First Down	Second Down	Third Down	First Up	Second Up	Third Up	Fourth Up
1	1	1	1	3	6	6
3	3	3	3	6	3	3
5	5	5	6	1	1	1
2	6	6	2	2	2	2
6	2	2	5	5	5	5
4	4	4	4	4	4	4

Figure 6. Rearranged Tables

Starting with the first point, number 1, it must be placed in the new table. From the table in Figure 5, the coupling terms for point 1 are found to be 3 and 5. Since neither of these terms are in the new table, they are added. The next point is 2 which does not appear in the new table and therefore, must be added. The coupling term for point 2 is 5 which already appears in the new table. The following point is 3 which already appears. Points 1 and 6 are the coupling terms for 3. Since point 1 already appears, the only addition to the table is point 6. The next point to be added to the table is point 4. No farther points will be added to the table since all the point numbers already appear. The second down column on the left of Figure 6 is developed using the same rules but instead of taking the points in numerical sequence, the points are taken in order from the first column in Figure 6. This process is repeated until the latest column matches the previous one. In the example problem, the third down column matches the second down column and this step of the process is terminated.

The next step is essentially a continuation of the previous step. The major difference is that the new table is started at the bottom and proceeds to the top and the points are taken from the previous table starting at the bottom and working to the top. This process is continued until a new table matches a previous one. In the example, this procedure was stopped after the fourth up table as shown in Figure 6.

The control matrix is then rearranged by column and row interchanges to match the last table. The resulting control matrix would appear as shown in Figure 7. No additional submatrices are required for the solution with this arrangement. However, before rearrangement, 8 additional submatrices were required for solution. Since the rearrangement was performed at the control matrix level, the actual quantity of numbers moved was small. In addition to rearranging the control matrix, it is also necessary to rearrange the other matrices associated with the solution of the equations. For instance, the B matrix in the example in Figure 1 must also be

	6	3	1	2	5	4
6	14	13				
3	12	4	3			
1		2	1		15	
2				5	6	
5			16	7	8	11
4					10	9

Figure 7. Resulting Control Matrix

rearranged when the control matrix for the A matrix is rearranged. The resulting Y matrix will also be in a rearranged order and will require interchanges to put it in correct order.

Through the use of partitioning with a control matrix and a rearrangement technique, it is possible to accomplish the following improvements. First, the time required to solve a set of equations can be reduced by a large factor. Secondly, with a given storage on a computer, the number of simultaneous equations that can be solved will be greater. Finally, the combination of time savings and greater capacity makes possible the solution of problems which were previously impossible.

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5

A Two Stage Inventory Model Applicable to a Provisioning Problem

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A rule is presented here under which two-stage inventory decisions can be made in the face of uncertainty. A common situation to which this rule applies is the one in which a buyer must determine the correct amount to be purchased and stocked of some commodity.

The article to be stocked is assumed to be procurable in both fully-finished and partially-finished form. There are some obvious applications to the stocking of spare service parts where the buyer's decision is made at the end of a production run.

In developing the rule, we assume that the decision maker has available to him

1. A forecast.
2. Reliable measures of the relevant costs of manufacturing and of obsolescence.
3. Estimate the value of customer good will (returned or lost).

The method of developing the rule to maximize expected profit follows the classical derivation of a single-stage inventory model with uncertainty. We extend the method, however, to two stages of inventory, i.e., partially finished and finished parts.

Our forecast of demand must be explicitly in the form of the cumulative distribution function, F(n), which describes the probability that the final outcome (sales) will be equal to or less than N. When the distribution function is absolutely continuous,[1] the frequency function and the cumulative distribution function are related by the expression

$$F(N) = \int_{-\infty}^{N} f(n)dn$$

[1] The assumption of an absolutely continuous distribution function is a good approximation for most economic situations.

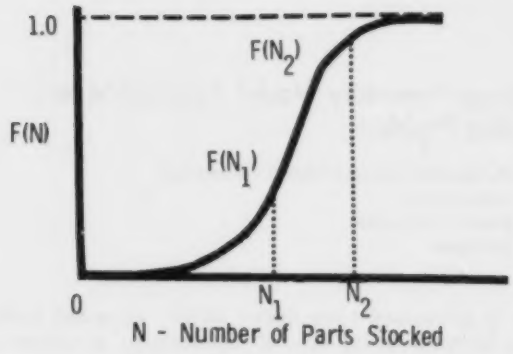

Figure 1.

Let:

c = current cost of finished part per unit

b = cost of partly finished part

a = cost to convert partially finished parts to finished parts at some future date (i.e., after the production run is terminated)

R = selling price per unit

M = good will of satisfied customer

N_1 = optimum number of finished pieces stocked

$N_2 - N_1$ = optimum number of partially finished pieces stocked.

Then N_1 and N_2 are given by

$$F(N_1) = \frac{a + b - c}{a}$$

$$F(N_2) = \frac{(R + M) - a - b}{(R + M) - a}$$

provided the following inequalities are satisfied:

$$R + M > a + b > c > b > o$$

and

$$\left(1 - \frac{b}{c}\right)(M + R) > a$$

The method of derivation is to determine the change-in-expected-profit resulting from the purchase of an additional finished piece and to compare this change-in-expected profit with that resulting from the purchase of an additional partially finished piece. Each time we consider buying an additional piece, we select that action which gives the greater increase in expected profit. Thus additional finished pieces or partially finished pieces are added to stock until purchase of neither will increase profit. The change-in-expected-profit with the number of pieces purchases (both for partially-finished pieces and finished pieces) is linear in the cumulative distribution of demand, F(N).

The case of principal interest is shown in Figure 2 in which we plot the change-in-expected-profit vs. F(N) for both finished and partially-finished piece purchases. The change-in-expected-profit for added finished pieces is

$$\frac{\partial \bar{P}}{\partial N_1} = (R + M - c) - (R + M)\ F(N_1) \tag{1}$$

and the change-in-expected-profit for added partially-finished pieces is

$$\frac{\partial \bar{P}}{\partial N_2} = (R + M - a - b) - (R + M - a)\ F(N_2) \tag{2}$$

For Figure (2), it has been assumed that

$$R + M - c > R + M - a - b$$

or

$$a + b > c \tag{3}$$

Inequality (3) means that the part which is bought now in finished form is cheaper than the partially finished part which is finished at some later date. If this inequality does not hold, all purchases should clearly be in partially finished pieces unless a = o (no cost of finishing) and b = c. In the latter case, there is no difference (for the purpose of this discussion) between purchase of finished parts and partial parts for later finishing. It has further been assumed in Figure 1 that

$$R + M > a + b \tag{4}$$

and

$$a > o \tag{5}$$

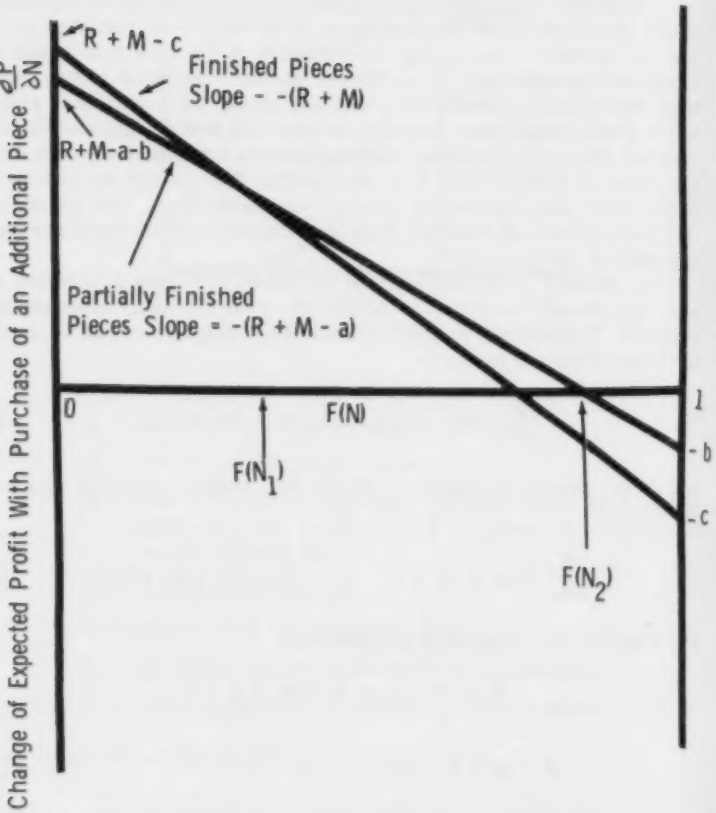

Figure 2.

Inequality (4) means that the total return from a sale (dollars plus good will) exceeds the cost of the item sold when partial parts are stocked for later finishing. If (4) did not hold, there would be no economic justification for stocking of partially finished parts. Inequality (5) means that there is some cost in converting partial parts to finished parts.

Under the above assumptions, it can be seen from Figure 2 that purchase of finished pieces increases profit more than purchase of partial pieces until N_1 finished pieces are purchased. N_1 is determined from the equation

$$(R + M - c) - (R + M) F(N_1) = (R + M - a - b) - (R + M - a) F(N_1)$$

or

$$F(N_1) = \frac{a + b - c}{a} \tag{6}$$

and the optimum quantity of finished pieces does not depend on either revenue from a sale or the value of customer good will.

Quantities stocked in amount greater than N_1 should be partially finished pieces until additional stock no longer increases expected profit. As indicated in Figure 2, this occurs when the total quantity of pieces stocked (both as finished and partially finished pieces) is N_2, where N_2 is given by the equation

$$(R + M - a - b) - (R + M - a) F(N_2) = 0$$

or

$$F(N_2) = \frac{R + M - a - b}{R + M - a} \tag{7}$$

Thus under the assumption resulting in Figure 1, N_1 finished pieces should be stocked. In addition, $N_2 - N_1$ partially finished pieces should be stocked. N_1 and N_2 are determined from equations (6) and (7).

Even though the assumptions thus far expressed for Figure 2, namely

$$R + M > a + b > c > b > 0$$

are met, a case other than that shown in Figure 2 could arise. Assume that "R + M" and "c" are the same as in Figure 2, but that "b" is increased and "a" is decreased so that "b" approaches "c" while "a + b" remains constant. Such a case is shown in Figure 3. As was found in Figure 2, Figure 3 shows that initially purchase of an additional finished piece increases expected profit more than purchase of an additional partial piece, but eventually a point is reached at which partially finished pieces are a "better buy" than finished pieces.

However, for the case shown in Figure 3, purchase of partially finished pieces is a "better buy" than purchase of finished pieces only in the sense that it causes profit to decrease less rapidly. At quantities for which the purchase of partially finished pieces is more desirable than purchase of finished pieces, stock of either additional finished pieces or additional partially finished pieces will

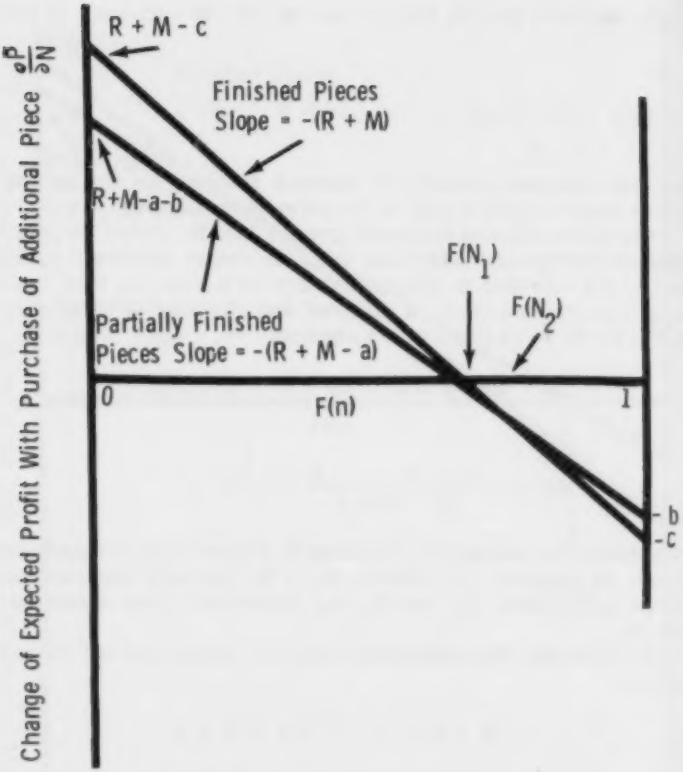

Figure 3.

decrease expected profit. Consequently, profit is maximized by stocking only finished pieces up to the quantity such that further purchases would decrease profit. Geometrically, this is the quantity corresponding to the value of the cumulative distribution, F(n), at which the $\frac{\partial \bar{P}}{\partial n}$ vs F(n) line crosses the F(n) axis. Analytically, this is given by

$$(R + M - c) - (R + M)\, F(N_1) = 0$$

or

$$F(N_1) = \frac{R + M - c}{R + M} \tag{8}$$

If the inequalities

$$R + M > a + b > c > b > 0$$

hold, the cases shown in Figures 2 and 3 are the only ones possible.

Case 1, the case of most practical interest, will arise if the finished part $\frac{\partial \bar{P}}{\partial n}$ vs F(n) line intersects the partially finished pieces $\frac{\partial \bar{P}}{\partial n}$ vs F(n) curve when $\frac{\partial \bar{P}}{\partial n}$ is positive. These two lines intersect at the point where

$$F(n) = \frac{a + b - c}{a}$$

so that Case 1 will apply if

$$(R + M - c) - (R + M)\frac{(a + b - c)}{a} > 0$$

or

$$a(R + M - c) + (R + M)(c - a - b) = (R + M)(c - b) - ac > 0$$

This reduces to

$$(1 - \frac{b}{c})(R + M) > a.$$

To complete the proof of the above statements, it is necessary only to show that the expressions for the change in profit with an additional purchase be those which have been used above. These may be established in a straightforward fashion as follows:

Let $P(N_1, n)$ = profit if N_1 finished pieces are purchased, given the demand is n.

The expected profit, $\bar{P}(N_1)$ is then given by

$$\bar{P}(N_1) = \int_0^\infty P(N_1 \mid, n)\, f(n)dn.$$

Now

$$P(N_1 \mid n) = Rn - cN_1 \quad 0 \leq n \leq N_1$$

And

$$P(N_1 \mid n) = (R - c)N_1 - M(n - N_1) \quad N_1 < n \leq \infty$$

Therefore

$$\bar{P}(N_1) = \int_0^{N_1} (Rn - cN_1)\, f(n)dn + \int_{N_1}^{\infty} [(R - c)N_1 - M(n - N_1)]\, f(n)dn$$

$$= \int_0^{N_1} Rn\, f(n)dn + \int_{N_1}^{\infty} RN_1\, f(n)dn - cN_1 + M \int_{N_1}^{\infty} (N_1 - n)\, f(n)dn$$

$$\frac{d\bar{P}(N_1)}{dN_1} = RN_1\, f(N_1) - RN_1\, f(N_1) + \int_{N_1}^{\infty} R\, f(n)dn - c - M(N_1 - N_1)\, f(N_1)$$

$$+ M \int_{N_1}^{\infty} f(n)dn$$

$$= (R + M) \int_{N_1}^{\infty} f(n)dn$$

$$= (R + M - c) - (R + M)\, F(N_1)$$

Similarly, let

$P(N_2, N_1 | n)$ = profit if a total of N pieces are stocked, N_1 finished and $(N_2 - N_1)$ partially finished given the demand is n.

Then

$$P(N_2, N_1 | n) = [Rn - cN_1 - b(N_2 - N_1)];\ 0 \le n \le N_1$$

$$= [Rn - cN_1 - b(N_2 - N_1) - a(n - N_1)];\ N_1 \le n \le N_2$$

$$= [RN_2 - cN_1 - (a + b)(N_2 - N_1) - M(n - N_2)];\ N_2 \le n \le \infty$$

Using these three expressions for the expected profit resulting if N_1 finished pieces are purchased and partially finished pieces for $(N_2 - N_1)$ additional pieces are purchased is given by

$$\bar{P}(N_2, N_1) = \int_0^{N_1} [Rn - cN_1 - b(N_2 - N_1)]\, f(n)dn$$

$$+ \int_{N_1}^{N_2} [Rn - cN_1 - b(N_2 - N_1) - a(n - N_1)]\, f(n)dn$$

$$+ \int_{N_2}^{\infty} [RN_2 - cN_1 - (a + b)(N_2 - N_1) - M(n - N_2)]\, f(n)dn$$

$$= \int_0^{N_2} Rn\, f(n)dn + \int_{N_2}^{\infty} RN_2\, f(n)dn - cN_1 - b(N_2 - N_1)$$

$$- \int_{N_1}^{N_2} a(n - N_1)\, f(n)dn - \int_{N_2}^{\infty} a(N_2 - N_1)\, f(n)dn$$

$$- M \int_{N_2}^{\infty} (n - N_2)\, f(n)dn$$

Therefore,

$$\frac{\partial \bar{P}(N_1, N_2)}{\partial N_2} = RN_2\, f(N_2) - RN_2\, f(N_2) + \int_{N_2}^{\infty} R\, f(n)dn - b$$

$$- a(N_2 - N_1)\, f(N_2) + a(N_2 - N_1)\, f(N_2) - \int_{N_2}^{\infty} a\, f(n)dn$$

$$+ M(N_2 - N_2)\, f(N_2) + \int_{N_2}^{\infty} M\, f(n)dn$$

$$= (R + M - a) \int_{N_2}^{\infty} f(n)dn - b$$

$$= (R + M - a)\,[1 - F(N_2)] - b$$

$$= (R + M - a - b) - (R + M - a)\, F(N_2)$$

Note that $\frac{\partial \bar{P}}{\partial N_2}$ does not depend upon the value of N_1, hence, the change in profit with additional purchase of partially finished pieces depend only upon the total of pieces purchased and not upon the way this total quantity is split between finished pieces and partially finished pieces. This fact made the graphical presentation given above possible.

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